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# Distribution-Sensitive Multidimensional Poverty Measures

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## Abstract

This paper presents axiomatic arguments to make the case for distribution-sensitive multidimensional poverty measures. The commonly-used counting measures violate the strong transfer axiom which requires regressive transfers to be unambiguously poverty-increasing and they are also invariant to changes in the distribution of a given set of deprivations amongst the poor. The paper appeals to strong transfer as well as an additional cross-dimensional convexity property to offer axiomatic justification for distribution-sensitive multidimensional poverty measures. Given the nonlinear structure of these measures, it is al also shown how the problem of an exact dimensional decomposition can be solved using Shapley decomposition methods to assess dimensional contributions to poverty. An empirical illustration for India highlights distinctive features of the distribution-sensitive measures.


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# Distribution-sensitive multidimensional poverty measures

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Recent estimates suggest that globally about 1.5 billion people live in multidimensional poverty (UNDP, 2015). Over the last decade or so, there has been a veritable explosion of literature on the measurement of multidimensional poverty. Following the initial Atkinson and Bourguignon (1982) paper and important later contributions of Tsui (2002), Bourguignon and Chakravarty (2003) and Alkire and Foster (2007, 2011a), the literature on both the theory and applications of multidimensional poverty measurement has grown rapidly.[1] With regards to applied work, measures within the “counting” approach (which focus on counting the number of dimensions in which people are deprived) have been influential, in particular the class of measures proposed by Alkire and Foster (2011a). One indicator of this influence is the publication of the Multidimensional Poverty Index (MPI) – which is a special case of the Alkire-Foster (hereafter AF) class of measures – for more than a 100 countries in the United Nations Development Program’s annual Human Development Report since 2010 (UNDP, 2010). This influence is only likely to grow with the recent adoption by the UN of the new set of Sustainable Development Goals (SDGs) which encompass a wide range of welfare dimensions many of which are also central to multidimensional poverty measurement.[2]

This paper revisits the measurement of multidimensional poverty from an axiomatic perspective, taking the AF measures as a point of departure. At its core, it presents axiomatic arguments for a class of “distribution-sensitive” measures. The arguments centre on the specific formulation of the dominance axioms[3], in particular, the transfer and the rearrangement axioms. The paper proposes stronger versions of these dominance axioms – a stronger version of the transfer axiom requiring regressive transfers (defined later) to be poverty-increasing, while also introducing a cross-dimensional convexity axiom requiring multiple deprivations to matter more than the sum of their parts, which is shown to be equivalent to a stronger version of the rearrangement axiom.

Two central ideas underlie this paper and the distribution-sensitive measures: (i) regressive transfers in any single dimension reduce social welfare, and (ii) multiple deprivations have compounding negative effects on individual and social welfare. The first has a long pedigree in the uni-dimensional poverty literature (see Sen, 1976, for instance) and there are no compelling reasons to eschew it in a multidimensional setting. The second is specific to the multidimensional context, and refers to the cumulative effects of multiple disadvantage on the severity of poverty, whereby the effect of disadvantage in one dimension is exacerbated by disadvantage in another. Multiple disadvantage can thus make it harder to graduate out of poverty. Recent evidence from Banerjee et al. (2015) on the success of multi-faceted interventions in tackling ultra-poverty lends support to the notion of multiple disadvantage as poverty traps. Hence, from this perspective, measures that disregard the extra burden of multiple deprivations could miss out on an important feature of the hardship experienced by those thus deprived.

The paper shows that poverty measures that are inclusive of *all* deprivations (union-based) and allow for the extra burden of multiple deprivations are distribution-sensitive in two respects: first, they satisfy the strong transfer axiom, and second, they are sensitive to inequality in the distribution of deprivations. The positive contribution of the paper is in bringing these two central ideas together to make a case for this class of distribution-sensitive multidimensional poverty measures which also satisfy a range of other desirable axioms proposed in the literature, without imposing any additional data requirements for practical applications. There are precursors to this class of measures in the literature in Bourguignon and Charavarty (2003), Chakravarty and D'Ambrosio (2006) and Alkire and Foster (2011a) – in that the particular mathematical functional form for the distribution-sensitive measures has been anticipated in these papers, though mainly in passing without exploring the special value of such measures in incorporating distributional considerations as discussed in this paper.

A property of multidimensional poverty measures that has been found to be attractive from a practical and policy perspective is that of dimensional decomposition (sometimes also referred to as factor decomposition), i.e. the ability to decompose the poverty measure into mutually exclusive and exhaustive contributions of each of the included dimensions. The AF measures, for instance, allow for a simple linear decomposition by dimension once the poor have been identified. However, this facility of linear decomposition is no longer available for distribution-sensitive poverty measures when they allow for compounding effects of deprivations across multiple dimensions. Another contribution of the paper is to show how an exact dimensional decomposition can be implemented in this case drawing upon the Shapley decomposition methods proposed by Shorrocks (2013). The Shapley decompositions are quite general and can be readily applied to any multidimensional poverty measure, including the AF measures. They also have the advantage of fully accounting for the contributions of dimensions both through the identification of the poor and the aggregation of their individual poverty levels.

The paper presents an application for India to illustrate key properties of distribution-sensitive poverty measures and how they compare with the AF class of measures. Besides presenting results on the implementation of Shapley dimensional decompositions, the application highlights several ways in which the use of distribution-sensitive measures could matter for measurement and analysis. First, it illustrates how much of multidimensional poverty could be potentially missed by measures that are not union-based and rely on a minimum number (or proportion) of deprivations to identify the poor. Second, it shows how the use of distribution-sensitive poverty measures can influence the profile of the poor and hence subgroup contributions to poverty. Third, it offers an example of how a maximal concentration of a given set of deprivations, while reflected in higher values of distribution-sensitive measures, can be consistent with a *fall* in counting measures. Fourth, using a "mean-inequality" type of decomposition, it shows how the contribution of inequality in the distribution of deprivations to distribution-sensitive measures could be assessed.

The paper is organized as follows. Section I introduces the multidimensional poverty measurement framework and the Alkire-Foster measures which offer the point of departure for this paper. Section II presents the strong transfer axiom, shows how it is violated by the AF class of measures, and elaborates the case for union-based multidimensional poverty measures that are inclusive of *all* deprivations. Section III raises the issue of the dispersion of deprivations across individuals to motivate the cross-dimensional convexity axiom. Section IV presents the distribution-sensitive multidimensional poverty measures and discusses their properties. Section V discusses Shapley dimensional decomposition. The application for India using data from the Third National Family Health Survey for 2005-6 is presented in section VI. The paper ends with some concluding remarks in section VII.

### I. The multidimensional framework and the AF measures

To facilitate presentation of the main arguments of this paper, we adopt largely the same notation as in Alkire and Foster (2011a). Thus, for a fixed population of $n$ individuals, $y = [y_{ij}]$ denotes an $n \times d$ matrix of non-negative achievements of individuals $i = 1, \dots n$ in dimensions $j = 1, \dots d$. The dimensional cut-offs (or "poverty lines" for each dimension) are denoted by $z_j > 0$ such that individual $i$ is deprived in dimension $j$ if $y_{ij} < z_j$, and non-deprived otherwise. Given $y$ and $z_j$, a 0-1 deprivation indicator function can thus be defined[4] as $I_{ij} = I(y_{ij} < z_j)$ and a corresponding $n \times d$ matrix of proportionate deprivations can be defined as $g^\alpha(y) = [g_{ij}^\alpha]$, where

$$g_{ij}^\alpha = \left(1 - y_{ij}/z_j\right)^\alpha I_{ij} \text{ for } \alpha \geq 0 \qquad (1)$$

An individual can be deprived in none, one or more dimensions. The number of deprived dimensions for individual $i$ is denoted by $c_i = \sum_{j=1}^{d} I_{ij}$, or equivalently, by the sum of the $i$-th row of $g^0(y)$. Thus, $c_i$ or $c_i/d$ could be interpreted as a measure of breadth of deprivations for an individual, while $g_{ij}^\alpha$ measures the depth or severity of individual deprivation in a specific dimension for $\alpha > 0$. For ease of exposition, we consider each dimension to carry an equal weight. The arguments presented below carry over to the more general case of unequal weights.

A key and innovative feature of the AF method is the idea of a dual cut-off to identify the poor. In addition to the dimensional cut-offs, AF define a poverty or cross-dimensional cut-off $k$ as the minimum number of dimensions in which an individual must be deprived in order to be deemed poor. Thus, $k$ lies between 1 and $d$, and individual $i$ is considered poor if $c_i \geq k$. Analogous and in addition to the deprivation indicator, a poverty indicator function can thus be defined as $I_i^k = I(c_i \geq k)$.

With this basic construct, Alkire and Foster (2011a) then define the following class of multidimensional poverty measures, $M(\alpha, k; y)$ as[5]:

$$M(\alpha, k; y) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{d}\sum_{j=1}^{d} g_{ij}^{\alpha}\right) I_i^k \qquad (2)$$

Later, it will also be useful to represent $M(\alpha, k; y)$ as an average of individual poverty measures, $m_i(\alpha, k; y_i)$:

$$M(\alpha, k; y) = \frac{1}{n}\sum_{i=1}^{n} m_i\,(\alpha, k; y_i) \text{ where } m_i(\alpha, k; y_i) = \left(\frac{1}{d}\sum_{j=1}^{d} g_{ij}^{\alpha}\right) I_i^k \qquad (3)$$

Using notation $\mu(..)$ to denote the mean of a matrix defined as the sum of all its elements of divided by the total number of its elements, $M(\alpha, k; y)$ can also be written as:

$$M(\alpha, k; y) = \mu(g^{\alpha}(k; y)) \text{ where } g^{\alpha}(k; y) = diag\left(I_i^k\right) g^{\alpha}(y) \qquad (4)$$

and $diag\left(I_i^k\right)$ is a $n \times n$ diagonal matrix with 1 or 0 as its diagonal elements according as person $i$ is poor (deprived in $k$ or more dimensions) or not. Note that $M(\alpha, k; y)$ can accommodate cardinal or ordinal data on achievements; if all achievements are binary, $M(0, k; y)$ is well-defined. Alkire and Foster (2011a) show that $M(\alpha, k; y)$ satisfies a number of axioms including: decomposability, replication invariance, symmetry, poverty and deprivation focus, weak monotonicity, dimensional monotonicity, non-triviality, normalization and weak rearrangement for $\alpha \geq 0$; monotonicity for $\alpha > 0$; and weak transfer for $\alpha \geq 1$.[6]

## II. Strong transfer axiom and the case for the union approach

It is now shown that $M(\alpha, k; y)$ violates an elementary form of the transfer axiom which, following Sen (1976), can be stated as the requirement that a transfer from a poorer to a richer person must increase measured poverty. While Sen referred to this axiom in the context of uni-dimensional poverty measures, it can be adapted to the multidimensional context using the following definition of a poorer person and a transfer in a deprived dimension $j'$. Person $i'$ is poorer than person $i''$ if $c_{i'} \geq k$ ($i'$ is poor) and $y_{i'j} \leq y_{i''j}$ for all $j$ and $y_{i'j'} < y_{i''j'}$ for at least one $j' \in \{1, 2, \ldots d\}$ ($i'$ has lower achievement than $i''$ in at least one dimension), or in other words, $y_{i''}$ dominates $y_{i'}$ ($y_{i''} D y_{i'}$). A (multidimensional) distribution $y^*$ is obtained from $y$ by a transfer in a deprived dimension $j'$ from individual $i'$ to $i''$ if $y_{i'j'} < z_{j'}$ and $y^*_{i'j'} = y_{i'j'} - \lambda y_{i'j'}$ and $y^*_{i''j'} = y_{i''j'} + \lambda y_{i'j'}$ for $0 < \lambda \leq 1$ , while $y^*_{ij} = y_{ij}$ for all other $(i, j) \neq \{(i'j'), (i'', j')\}$. The strong transfer axiom[7] in the multidimensional context can now be stated.

**Strong transfer axiom**: If $y^*$ is obtained from $y$ by a transfer in a deprived dimension from a poorer to a richer person, then $M(\alpha, k; y^*) > M(\alpha, k; y)$.

It is useful to distinguish the strong transfer axiom as stated above from the one-dimensional transfer principle (OTP) in Bourguignon and Chakravarty (2003). Despite their deceptive similarity, OTP is more demanding than the strong transfer axiom: the key difference is that OTP does not require $y_{i''}$ to dominate $y_{i'}$ but only $y_{i'j'} < y_{i''j'}$ for the transfer in dimension $j'$ to be poverty-increasing or at least poverty non-decreasing. There is however the issue that if $y_{i'j} > y_{i''j}$ for other dimensions $j \neq j'$, then in what respect should person $i'$ (the transferor) be considered “poorer” than person $i''$ (the transfer recipient). By requiring $y_{i''}$ to dominate $y_{i'}$, the transfer axiom makes the notion of “poorer”, and hence the notion of a “regressive” transfer, more precise.

The violation of the strong transfer axiom by $M(\alpha, k; y)$ is now demonstrated by an example. Consider the following 3 person 4 dimension case where the matrices of achievements before and after transfer (by person 2 of two-thirds of her achievement in the 3rd dimension to person 1) are given by:

$$y = \begin{bmatrix} 15 & 8 & \mathbf{70} & 0 \\ 3 & 4 & \mathbf{60} & 0 \\ 11 & 7 & 150 & 1 \end{bmatrix} \rightarrow y^* = \begin{bmatrix} 15 & 8 & \mathbf{110} & 0 \\ 3 & 4 & \mathbf{20} & 0 \\ 11 & 7 & 150 & 1 \end{bmatrix}$$

Let the dimensional cut-offs be $z = [10 \quad 5 \quad 100 \quad 1]$ and $k = 2$. Then, prior to the transfer, this 3-person community has two poor persons (viz., 1 and 2), and person 2 (the transferor) is unambiguously poorer than person 1 (the transfer recipient) as her achievement vector is dominated by person 1’s achievement vector. Post-transfer, person 1 becomes non-poor. The corresponding pre- and post-transfer deprivation matrices are as below:

$$g^\alpha(y) = \begin{bmatrix} 0 & 0 & (\mathbf{0.3})^{\boldsymbol{\alpha}} & 1 \\ (\mathbf{0.7})^{\boldsymbol{\alpha}} & (\mathbf{0.2})^{\boldsymbol{\alpha}} & (\mathbf{0.4})^{\boldsymbol{\alpha}} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow g^\alpha(y^*) = \begin{bmatrix} 0 & 0 & \mathbf{0} & 1 \\ (\mathbf{0.7})^{\boldsymbol{\alpha}} & (\mathbf{0.2})^{\boldsymbol{\alpha}} & (\mathbf{0.8})^{\boldsymbol{\alpha}} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is easy to check that $M(\alpha, k; y^*) < M(\alpha, k; y)$ for all values of $\alpha$. A regressive transfer in a single dimension can *reduce* poverty measured by the AF measure, thus violating the strong transfer axiom. The violation occurs even for $\alpha > 1$ because the transfer to the less poor person can not only eliminate a particular deprivation for that person, but it can also make the person non-poor if she was deprived in exactly $k$ dimensions prior to receiving the transfer. Extending the example further, it can be seen that if there is a significant mass of people deprived in $k$ dimensions, it will often be possible to steadily reduce the AF measure of multidimensional poverty by a series of regressive transfers from poorer persons, even to the point where achievements of these poorer transferors are driven down to zero in several dimensions.

Conversely, it is easy to check that for $\alpha > 1$ the violation will not occur if the recipient remains poor after the transfer. Fundamentally, the violation arises because the "dual cut-off" assigns a zero weight to any deprivations of non-poor persons.

The statement of the problem also suggests an immediate solution: make every deprivation count by setting $k = 1$ (or, more generally for unequally weighted dimensions, setting the poverty cut-off at no more than the minimum of dimensional weights). This is the so-called *union* approach to identification whereby a person is considered poor if deprived in at least one dimension. It yields the following multidimensional poverty measure:

$$M(\alpha; y) = \frac{1}{nd}\sum_{i=1}^{n}\sum_{j=1}^{d} g_{ij}^{\alpha} \tag{5}$$

This is the same as the additive multidimensional extension of the FGT (Foster, Greer, Thorbecke, 1984) measure in Bourguignon and Chakravarty (2003). While Alkire and Foster (2011a) present their dual cut-off measures as a generalization of the Bourguignon-Chakravarty measure, the generalization comes at a price (most directly, the violation of the strong transfer axiom as above) that we may not want to pay.

Alkire and Foster (2011a) briefly consider the union method, but conclude that it is not appropriate in all circumstances. They offer two arguments. First, "deprivation in certain single dimensions may be reflective of something other than poverty". Second, "when the number of dimensions is large, the union approach will often identify most of the population as being poor… Consequently, a union-based poverty methodology may not be helpful in distinguishing and targeting the most extensively deprived". For convenience, let's term these as the mismeasurement argument (what is measured is not "true" deprivation or poverty) and the targeting argument (what is measured is not helpful for targeting). On closer scrutiny, however, neither argument is particularly compelling.

In relation to the mismeasurement issue, it is arguable that the censoring implied by the dual cut-off method presents a rather amputative approach to dealing with measurement error, which has several problems. First, it assumes that *all* deprivations of an individual must be "fortuitous" or measured with error if the total number of deprivations for the individual falls short of the poverty cut-off. Second, measurement errors are unlikely to be discontinuous at deprivation thresholds such that it is also possible (on account of measurement error) for someone to be deemed non-deprived in a particular dimension when in fact she is.[8] With dual cut-off censoring, it is then possible for this individual to be deemed non-poor, and thus exacerbating the problem by not only ignoring her mismeasured deprivation, but also all her other deprivations too. Third, measurement errors are also likely to differ across dimensions: some dimensions are harder to measure accurately than others. Yet, dual cut-off censoring as a response to measurement error is completely non-specific with respect to dimensions. Fourth, dual cut-off as a response to measurement error also carries the problematic implication that measurement error matters for those with fewer deprivations than the cross-dimensional cut-off, but may be ignored for those with more deprivations. For all these reasons, the dual cut-off approach does not appear to be a methodologically appropriate response to measurement error.

In relation to the targeting argument, it is useful to note that given subgroup decomposability, the union-based poverty measure can be written as population share ($s_j$)-weighted sum of poverty measures of $d$ mutually exclusive groups comprising respectively of those who are deprived in exactly $j$ dimensions for $j = 1, 2, \ldots d$.

$$M(\alpha; y) = s_1 M(\alpha; y|c_i = 1) + s_2 M(\alpha; y|c_i = 2) + \cdots + s_d M(\alpha; y|c_i = d) \qquad (6)$$

Thus, if budgetary resources or other considerations do not allow policymakers to target the full set of those facing *any* deprivation, it is always possible to drill down a union-based poverty measure and focus on subsets of those deprived in multiple dimensions. For instance, one could start with those deprived all $d$ dimensions, then move on to those deprived in $d - 1$ dimensions, and so on. Alternatively, it is possible to define a target group as bottom $x\%$ of the population when individuals are ranked from the poorest to the least poor in terms of their individual poverty levels $m_i(\alpha; y)$. The need to identify a target group does not by itself offer adequate justification for the use of a cross-dimensional cut-off.[9]

As against this, one needs to weigh in the potential benefits of the union approach, a significant one of which is the satisfaction of the strong transfer axiom above. In addition, the union-based approach also allows the monotonicity and dimensional monotonicity axioms to be strengthened as below.

**Strong monotonicity**[10]: If distribution $y^*$ is obtained from $y$ by a decrement in any deprived dimension $j'$ for any individual $i'$ [i.e. there exists an $(i', j')$ such that $y^*_{i'j'} < y_{i'j'} < z_{j'}$ while $y^*_{ij} = y_{ij}$ for all other $(i, j) \neq (i', j')$], then $M(\alpha; y^*) > M(\alpha; y)$.

Note that strong monotonicity implies monotonicity which only requires the poverty-increasing property of a decrement in achievements to be satisfied if such decrement occurs for those deprived in at least $k$ dimensions. Conversely, for $\alpha > 0$, strong monotonicity will be violated by the AF class of measures unless $k = 1$, i.e., the union case. A further implication of strong monotonicity is also notable: strong monotonicity implies strong dimensional monotonicity.

**Strong dimensional monotonicity**: If distribution $y^*$ is obtained from $y$ by an increment in a deprived dimension $j'$ for any individual $i'$ which makes her non-deprived in that dimension, [i.e. there exists an $(i', j')$ such that $y^*_{i'j'} \geq z_{j'} > y_{i'j'}$ while $y^*_{ij} = y_{ij}$ for all other $(i, j) \neq (i', j')$], then $M(\alpha; y^*) < M(\alpha; y)$.

Strong dimensional monotonicity implies dimensional monotonicity (as in Alkire and Foster, 2011a) where the latter restricts the application of the poverty-decreasing property of the removal of a given deprivation to only those deprived in at least $k$ dimensions. Strong dimensional monotonicity will be generally violated by the AF class of measures whenever the elimination of a particular deprivation occurs for someone who is deprived in less than $k$ dimensions. Such violation is precluded by setting $k = 1$.

Does the violation of strong monotonicity properties matter? A reason for preferring the stronger versions is that they imply the weaker versions but not the converse. The key difference between the two (as noted above) is that the weaker versions restrict monotonicity to only those who are deprived in at least $k$ dimensions, while the stronger versions apply monotonicity to those deprived in *any* dimension. The difference thus comes down to the issue of essentiality of all deprivations. If all deprivations are deemed worthy of inclusion in a poverty measure, as argued above, then it follows that stronger monotonicity properties are appropriate. A decrement in an inclusion-worthy deprived dimension *should* lead to an increase in measured poverty, rather than no change; weaker versions can end up ignoring the impact of some deprivations.

## III. Sensitivity to concentration of deprivations and the idea of cross-dimensional convexity

Another potential issue with the AF class of counting measures is illustrated by returning to our 3-person 4-dimension example. Consider the following two deprivation matrices with $\xi_j \gg 0$ and $k = 2$:

$$g^{\alpha}(y) = \begin{bmatrix} (\boldsymbol{\xi_1})^{\boldsymbol{\alpha}} & (\boldsymbol{\xi_2})^{\boldsymbol{\alpha}} & 0 & 0 \\ 0 & 0 & (\boldsymbol{\xi_3})^{\boldsymbol{\alpha}} & (\boldsymbol{\xi_4})^{\boldsymbol{\alpha}} \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow g^{\alpha}(y^*) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ (\boldsymbol{\xi_1})^{\boldsymbol{\alpha}} & (\boldsymbol{\xi_2})^{\boldsymbol{\alpha}} & (\boldsymbol{\xi_3})^{\boldsymbol{\alpha}} & (\boldsymbol{\xi_4})^{\boldsymbol{\alpha}} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, achievement matrices $y$ and $y^*$ are associated with the same combination of deprivations, but while in $y$ individuals 1 and 2 share two deprivations each, in $y^*$ all four deprivations are experienced by individual 2. Since the means of the two deprivation matrices are the same, $M(\alpha, k; y) = M(\alpha, k; y^*)$ for all values of $\alpha$. Thus, the AF class of measures are completely insensitive to how a given set of deprivations is spread across the poor; maximal and minimal concentrations of deprivations are treated alike.

The above example illustrates a case where (in moving from $y$ to $y^*$) there is a change in the joint distribution of achievements but no change in the censored marginal distributions. Note however that the invariance of AF measures to such changes in joint distribution relates to $M(\alpha, k;\ y)$ measures, and not to partial indices, such as $H$ and $A$, which indeed do not satisfy many of the (axiomatic) properties proposed for the AF measures. For instance, it is true that in the example above, in moving from distribution $y$ to $y^*$, while poverty incidence, $H$, is halved, the intensity of poverty amongst the poor, $A$, doubles, which is why the product $HA = M(0, k; y)$ remains unchanged. But that only highlights the key issue. As the halving of $H$ is a desirable change and the doubling of $A$ is the opposite, how do we form a judgement on the overall change? The judgement implicit in the AF measures $M(\alpha, k; y)$, that these two effects always exactly cancel each other out, is contentious. As argued below, the distribution-sensitive $M(\alpha, \beta; y)$ measures for $\beta > 1$ incorporate the alternative judgement that the same set of deprivations if shared more (less) equally imposes a lesser (greater) burden on society.

Such distributional invariance of AF measure, ironically, renders them somewhat similar *in that respect* to indices such as the Human Development Index (HDI) or the Human Poverty Index (HPI) which depend only on the marginal distributions of indicators they aggregate.[11] Strictly, the joint density across dimensions matters for the AF measures at the identification stage (in determining whether someone is poor or not), but not at the aggregation stage. Put differently, the AF measures depend on the censored marginal distributions, and that renders them insensitive to greater or lesser concentration of a given set of deprivations amongst the poor so long as the set of poor remains the same.

It is notable that this issue persists even in the union case when $k = 1$ and there is no censoring. In general, $M(\alpha; y)$ will be unchanged for any rearrangement of a *given* set of deprivations for all values of $\alpha$. In this respect, the AF class of measures (including the special union case with $k = 1$) are rather utilitarian in their reliance on a "sum-ranking"[12], i.e., the aggregation of individual deprivations by summing them up.

It is arguable however that a given deprivation is more burdensome for an individual if also accompanied by deprivation in other dimensions. A similar view is echoed by Stiglitz, Sen and Fitoussi (2009, p. 15) in their discussion of measures of the quality of life: "…the consequences for quality of life of having multiple disadvantages far exceed the sum of their individual effects". The notion of compounding negative effects of multiple disadvantages is also deeply rooted in the sociological concept of the *underclass* (e.g. Wilson, 1987, 2006), in policy discussions of social exclusion (e.g. Levitas et al, 2007), as well as in some formulations of "poverty traps" (e.g. Durlauf, 2006; Banerjee and Duflo, 2011). Common to these diverse perspectives is the idea that through various mechanisms multiple disadvantages can reinforce each other in the reproduction of poverty for certain groups, in undermining the long-term "life chances" or future prospects of the most disadvantaged in society.[13]

It is thus eminently arguable that the concentration or dispersion of deprivations across individuals matters, and a case can be made for the requirement that multidimensional poverty measures be sensitive to the distribution of a given set of deprivations. This can be achieved by requiring the *individual* multidimensional poverty measure to be convex over individual deprivations in different dimensions, and can be axiomatized as follows.

**Cross-dimensional convexity**: An increment in deprivation $j'$ for individual $i$ induces a greater increase in measured poverty for individual $i$, the greater individual $i$'s deprivation in any other dimension $j \neq j'$.[14]

There is no real mystery to the idea that if you are hurting on many fronts, an additional blow could be completely crushing. The cross-dimensional convexity axiom appeals to this idea of the compounding (negative) effect of multiple deprivations on the overall well-being of the individual. It asserts that the severity of the effects of an increase in deprivation in any one dimension increases not only with the level of deprivation in that dimension but also with the level of deprivation in other dimensions. Put differently, it is a way of capturing the added welfare "cost" of an additional deprivation, where the added cost itself depends on the extent of deprivation in other dimensions.

Additional support for cross-dimensional convexity comes from Duclos, Sahn and Younger (2006) who derive stochastic tests for robust multidimensional poverty comparisons using the assumption of positive cross-partial derivatives of the individual poverty function with respect to dimensional achievements, and show that the scope for robust poverty orderings would be drastically limited without positive cross-partial derivatives, which is equivalent to cross-dimensional convexity for continuous poverty functions.

### IV. Distribution-sensitive poverty measures

The cross-dimensional convexity axiom suggests the following general class of what may be called "distribution-sensitive" measures:[15]

$$M(\alpha, \beta; y) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{d}\sum_{j=1}^{d} g_{ij}^{\alpha}\right)^{\beta} \quad \text{for } \alpha \geq 0 \text{ and } \beta \geq 1 \tag{7}$$

Note that $M(\alpha, \beta; y)$ can be written as an average of individual poverty measures, $m_i(\alpha, \beta; y_i)$:

$$M(\alpha, \beta; y) = \tfrac{1}{n}\textstyle\sum_{i=1}^{n} m_i\,(\alpha, \beta; y_i) \quad \text{where } m_i(\alpha, \beta; y_i) = \left(\tfrac{1}{d}\textstyle\sum_{j=1}^{d} g_{ij}^{\alpha}\right)^{\beta} \tag{8}$$

The measure $M(\alpha, \beta; y)$ could be considered a generalization of the union-based measure $M(\alpha; y)$ discussed earlier, the latter obtained for the special case when $\beta = 1$. It is readily verified that for values of $\beta > 1$, the measure $M(\alpha, \beta; y)$ satisfies the cross-dimensional convexity axiom. The value of $\beta$ can be interpreted as parameterizing the relative weight accorded to the multiplicity of deprivations, i.e., to the joint density of deprivations relative to the marginal distributions of single deprivations.[16] Besides cross-dimensional convexity, $M(\alpha,\beta;y)$ also satisfies a number of other axioms as summarized in Proposition 1.

***Proposition 1****: The multidimensional poverty measure $M(\alpha,\beta;y)$ satisfies deprivation focus, poverty focus, subgroup decomposability, replication invariance, symmetry, normalization, strong dimensional monotonicity for $\alpha \geq 0, \beta \geq 1$, strong monotonicity for $\alpha > 0$ and $\beta \geq 1$, cross-dimensional convexity and strong rearrangement for $\alpha \geq 0, \beta > 1$, and strong transfer for $\alpha > 1, \beta \geq 1$ or $\alpha \geq 1, \beta > 1$.*

Formal statements of these axioms and a proof of Proposition 1 are given in the Supplemental Appendix.[17] Several remarks can be made on the properties of the distribution-sensitive measure $M(\alpha,\beta;y)$. However, before turning to these, it is useful to comment on some antecedents to this measure in the literature. Four are notable.

First, when deprivations are measured in only ordinal binary form, thus precluding measurement of the depth of deprivations, the measure $M(\alpha,\beta;y)$ reduces to:

$$M(\alpha,\beta;y) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{c_i}{d}\right)^{\beta} \quad \text{for } \beta \geq 1 \qquad (9)$$

and coincides with the measure of social exclusion in Chakravaty and D'Ambrosio (2006). Thus, $M(\alpha,\beta;y)$ could be viewed as a generalization of the Chakravaty-D'Ambrosio social exclusion measure. As the social exclusion measure is restricted to deprivations in binary terms, the strong transfer axiom is not applicable. However, in the binary case too, the $M(\alpha,\beta;y)$ measure provides an important means of incorporating "inequality aversion" into multidimensional poverty measurement. By comparison, the AF measure in this case simplifies to

$$M(\alpha, k; y) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{c_i}{d}\right)I_i^k = \frac{\bar{\bar{c}}}{d} \tag{10}$$

and depends only on the average number of censored deprivations, not their distribution across individuals.[18]

Second, one of the non-additive measures[19] in Bourguignon and Chakravarty (2003), also discussed in Atkinson (2003), takes the form:

$$\frac{1}{n}\sum_{i=1}^{n}\left(a_1 g_{i1}^{\theta} + a_2 g_{i2}^{\theta}\right)^{\lambda/\theta}$$

in a two-dimensional context. It is readily seen that this measure has the same mathematical form as $M(\alpha, \beta; y)$ in (7) for $\theta = \alpha$ and $\lambda = \alpha\beta$. It is also readily checked that the Bourguignon-Chakravarty measure will satisfy the strong transfer axiom if $\lambda = \theta > 1$ or if $\lambda > 1$ and $\theta = 1$, but in general will fail to satisfy it if $\lambda/\theta < 1$. However, Bourguignon and Chakravarty simply present $\lambda > \theta$ or $\lambda < \theta$ as alternative possibilities, while from the perspective of this paper, it is not a matter of normative indifference whether $\lambda/\theta$ is greater or less than one. The case of $\lambda < \theta$ (or $\beta < 1$ in (7) above) would not only violate cross-dimensional convexity but also the strong transfer axiom and monotonicity properties as suggested in the paper.

Third, the distribution-sensitive measure is also anticipated in the concluding section of Alkire and Foster (2011a, p. 485) where they observe that one could “replace the individual poverty function $M_\alpha(y_i; z)$ with $[M_\alpha(y_i; z)]^\gamma$ for some $\gamma > 0$ and average across persons”, also noting that Bourguignon and Chakravarty (2003) present similar measures. But like Bourguignon and Chakravarty, they treat $\gamma < 1$ and $\gamma > 1$ as alternative possibilities, which they note requires dimensions to be all substitutes or all complements, and with an equal strength across dimensions. They consider this to be “unduly restrictive” and hence present $\gamma = 1$ as their “basic neutral case”. However, since like Bourguignon and Charkarvarty (2003), Alkire and Foster (2011a) stop short of presenting anything like the cross-dimensional convexity axiom (which accords normative primacy to the $\gamma > 1$ case), the distribution-sensitive measure $M(\alpha, \beta; y)$, though anticipated by Alkire and Foster, has hitherto lacked axiomatic justification. In addition, as already noted above, the Alkire-Foster measures on account of their use of the dual cut-off violate the strong transfer axiom.

Fourth, it is also worthwhile commenting briefly on a related set of "inequality-sensitive" multidimensional poverty measures in Rippin (2012, 2013, 2016) which, for uniform dimensional weights, take the form[20]

$$M(\alpha, \kappa; y) = \frac{1}{n}\sum_{i=1}^{n}(c_i/d)^{\kappa}\sum_{j=1}^{d} g_{ij}^{\alpha} \text{ for nonnegative } \alpha, \kappa \tag{11}$$

Motivated by considerations similar to this paper, these measures explicitly build in the breadth of individual deprivations $(c_i/d)$ and are therefore sensitive to greater or lesser concentration of deprivations amongst the poor. However, it can be shown that though less likely to violate the strong transfer axiom than the AF measures (on account of union-based identification and the extra weight accorded to the breadth of deprivations), the $M(\alpha, \kappa; y)$ measures do not preclude the violation even for values of $\alpha, \kappa > 1$. This is because while $\alpha > 1$ ensures that in case of a regressive transfer in a deprived dimension $j$, the reduction in deprivation $j$ for the recipient is no greater than the increase in deprivation $j$ for the giver, it is possible that the recipient becomes non-deprived in $j$ as a result of the transfer, and the consequent decline in her deprivation breadth (for $\kappa \geq 1$) lowers the weight on her remaining deprivations sufficiently for her individual poverty to decline by a greater amount than the increase in individual poverty of the giver.

Thus, as an overall comment on the antecedent literature, one may note that while similar-looking measures have appeared in this literature, it has not explored the *joint* satisfaction of the strong transfer and cross-dimensional convexity axioms, which together constitute the key motivating properties of the distribution-sensitive measures. The rest of this section offers some further explanatory remarks on the properties of distribution-sensitive measure $M(\alpha,\beta;y)$.

**(i)** Unlike the AF class of measures, $M(\alpha,\beta;y)$ satisfies the strong transfer axiom for $\alpha > 1, \beta \geq 1$ or $\alpha \geq 1, \beta > 1$ (see ***1i*** in Supplemental Appendix for a proof). In this regard, $M(\alpha,\beta;y)$ is also different from the "non-additive" measures in Bourguignon and Chakravarty (2003), which in general satisfy the multidimensional transfer principle (MTP), which is equivalent to the weak transfer axiom in Alkire and Foster (2011a). Bourguignon and Chakravarty (2003) however do not consider whether their non-additive measures satisfy the strong transfer axiom as stated in this paper, which (as noted in section II) is less demanding than their one-dimensional transfer principle (OTP). Based on their Proposition 3, they conclude that all non-additive measures do not satisfy OTP. However, there do exist non-additive measures that, while they violate OTP, nonetheless do satisfy the strong transfer axiom as stated above. Indeed, $M(\alpha,\beta;y)$ is an example of such a non-additive measure.

**(ii)** $M(\alpha,\beta;y)$ also satisfies a stronger version of the (weak) rearrangement axiom in Alkire and Foster (2011a). This axiom considers a switch of achievements between two poor persons such that prior to the switch one person's achievement vector dominates the other's, but such vector dominance no longer obtains after the switch. Such a switch is sometimes referred to as an association-decreasing switch. The strong rearrangement axiom requires that such a decrease in inequality (due to an association-decreasing switch) is poverty-reducing as long as the switch involves a deprived dimension for one of them and so long as it is not a trivial switch. A trivial switch is one where switched achievements are identical.

**Strong rearrangement axiom**. If for an initial distribution of achievements $y$, $y_{i'}Dy_{i''}$ (i.e. $y_{i'}$ dominates $y_{i''}$) and both $i'$ and $i''$ are poor (i.e. deprived in at least one dimension in the union case with $c_{i''} \geq c_{i'} > 0$), the distribution $y^*$ is obtained by a non-trivial switch of achievement $j'$ between $i'$ and $i''$ [i.e. $y^*_{i'j'} = y_{i''j'}$ and $y^*_{i''j'} = y_{i'j'}$ while $y^*_{ij} = y_{ij}$ for all other $(i,j) \neq \{(i'j'),(i'',j')\}$ ] such that $y^*_{i'} \sim D y^*_{i''}$ (i.e. $y^*_{i'}$ does not dominate $y^*_{i''}$), then $M(\alpha,\beta;y^*) < M(\alpha,\beta;y)$ unless $j'$ is not a deprived dimension for $i''$ in $y$,

The AF measure satisfies a weaker version of this axiom with $M(\alpha,k;y^*) \leq M(\alpha,k;y)$ (see Supplemental Appendix for a definition of the weak rearrangement axiom), but the weaker version is *always* satisfied in equality, i.e. $M(\alpha,k;y^*) = M(\alpha,k;y)$ for any such switch of achievements between two poor persons. In other words, $M(\alpha,k;y)$ measure cannot decline for the kind of decrease in inequality implied by an association-decreasing switch of achievements. By virtue of cross-dimensional convexity, $M(\alpha,\beta;y)$ measure for $\beta > 1$, on the other hand, will allow a decline in measured poverty if the switch between $i'$ and $i''$ involves a deprived dimension for both individuals with $y_{i'j'} > y_{i''j'}$ (i.e. $g^{\alpha}_{i''j'} > g^{\alpha}_{i'j'} > 0$) or a deprived dimension for one of them ($g^{\alpha}_{i''j'} > 0$ and $g^{\alpha}_{i'j'} = 0$); see ***1k*** in Supplemental Appendix for a proof.

The switch considered in the rearrangement axioms also admits the possibility of a reverse switch (switching back) where $y_{i'}$ does not dominates $y_{i''}$ before the switch, but it does so after the switch (i.e. $y_{i'} \sim D y_{i''}$ but $y^*_{i'} D y^*_{i''}$). The rearrangement axiom in this case requires poverty to increase, $M(\alpha,\beta;y^*) > M(\alpha,\beta;y)$, if the switch involves a deprived dimension for one or both individuals with $g^{\alpha}_{i''j'} > g^{\alpha}_{i'j'} \geq 0$. Thus, as before, $M(\alpha,\beta;y)$ allows an association-increasing switch to be poverty increasing while this is not permitted by $M(\alpha,k;y)$.

**(iii)** It is not an accident that cross-dimensional convexity and strong rearrangement are satisfied for $\beta > 1$ in Proposition 1. The two properties are equivalent: cross-dimensional convexity implies strong rearrangement and vice versa; see Supplemental Appendix for a proof. The satisfaction of the strong rearrangement axiom and cross dimensional convexity for $\beta > 1$ in turn ensures that a greater concentration of a given set of deprivations amongst fewer individuals will increase poverty measured by the distribution-sensitive measure $M(\alpha,\beta;y)$.

**(iv)** The satisfaction of cross-dimensional convexity for $\beta > 1$ can be interpreted as asserting *complementarity* of multiple deprivations in different dimensions. There has been some confusion in terminology regarding this in the literature. The axiom of cross-dimensional convexity implies a *complementarity between different dimensional shortfalls* for an individual in the sense that a person's multidimensional poverty $m_i$ increases by a larger amount in response to increased deprivation in one dimension ($g_{ij}$), the greater this person's deprivation in another dimension ($g_{ij'}$). That is, (for continuous functions) the cross-partial derivative of individual poverty to dimensional deficits is positive. But since

$$\frac{\partial^2 m_i(\alpha,\beta;y)}{\partial y_{ij}\partial y_{ij'}} = \frac{1}{z_j z_{j'}} \frac{\partial^2 m_i(\alpha,\beta;y)}{\partial g_{ij}\partial g_{ij'}} \ ,$$

it follows that a positive cross-partial derivative with respect to dimensional deficits also implies a positive cross-partial derivative with respect to dimensional achievements. However, the latter is what Bourguignon and Chakravarty (2003) refer to as the dimensions being substitutes. Whether we call them complements or substitutes, what matters for cross-dimensional convexity is $\frac{\partial^2 m_i(\alpha,\beta;y)}{\partial g_{ij}\partial g_{ij'}}$ being positive. The key point is that however one may want to think of complementarity or substitutability of dimensions on the production side (how one attribute or dimension is transformed into another), *when there is a shortfall in multiple dimensions*, cross-dimensional convexity asserts that it hurts more than the sum of single dimensional shortfalls. And it is in this precise sense that under cross-dimensional convexity the shortfalls in different dimensions are deemed complementary.[21]

The issue of whether production complementarities between attributes pose a challenge to the normative appeal of a property like strong rearrangement or cross dimensional convexity has featured frequently in the literature in various forms; for instance, the argument is made that considerations of complementarity between attributes should make room for departures from cross-dimensional convexity.[22] One can discuss the key issue at two levels: (i) in the stark case of perfect complementarity between two attributes in the specific sense that the possession of one attribute (e.g. refrigerators) is of no use without the possession of the other (e.g. electricity); (ii) in the context of the more general and larger question of whether or how complementarity should influence multidimensional poverty measurement.[23] In the first case, cross-dimensional convexity (CDC) would seem to favour a more even distribution of electricity connections and refrigerators, even though refrigerators without electricity of no use (abstracting from the possibility that they can be sold for something else). However, a reasonable response here could be to redefine the two underlying deprivations to take into account this particular feature of the dimensions being considered. Thus, in the specific example, the two deprivations could be redefined as: one, not having access to electricity, and two, having access to electricity <u>and</u> not having a refrigerator. With this redefinition, handing out refrigerators to those without an electricity connection will not serve to reduce the second deprivation, opening the door for a consistent application of CDC.

Turning to the more general issue of complementarity between attributes, one can argue with some justification that how production complementarities between attributes (how one attribute is transformed into another) should inform policy choices is better handled independently of how a social welfare function is specified (viewing multidimensional poverty measures as social welfare functions). Trying to incorporate such production complementarities into the specification of social welfare functions on the other hand can be confounding. One can illustrate this point with the following example. Consider two attributes, nutrition (N) and education (E) and two persons, A and B, with the following deprivation matrix, $g'$:

$$g' = \begin{pmatrix} & E & N \\ A & 0.8 & 0.2 \\ B & 0.6 & 0.6 \end{pmatrix}$$

Person A has an education deficit of 80% of the education threshold and nutrition deficit of 20% of the nutrition threshold, while person B has a 60% deficit of both attributes. Thus, A has a higher education deprivation while B has a higher nutrition deprivation. Now consider a correlation-increasing switch of the nutrition deprivation resulting in $g''$:

$$g'' = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} E & \quad N \end{array} \\ \begin{pmatrix} 0.8 & 0.6 \\ 0.6 & 0.2 \end{pmatrix} \end{array}$$

What the cross-dimensional convexity (CDC) axiom asserts in this case is that poverty in $g''$ should be higher than in $g'$. Thus, $CDC \Rightarrow M(g'') > M(g')$. Let us now invoke a production complementarity where better nutrition leads to better education outcomes such that with lower nutrition deficit after the switch, B's education deficit also declines, say, to 0.2. By the same token, A's higher nutrition deficit after the switch increases his education deprivation, say to 0.9. Thus, if we build in production complementarity, in effect we should be comparing $g'$ with the following deprivation matrix, $g'''$ after the switch:

$$g''' = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} E & \quad N \end{array} \\ \begin{pmatrix} 0.9 & 0.6 \\ 0.2 & 0.2 \end{pmatrix} \end{array}$$

However, in comparing $g'$ with $g'''$, CDC (or the strong rearrangement axiom) does <u>not</u> assert that poverty in $g'''$ should be necessarily higher than in $g'$. Thus, $CDC \nRightarrow M(g''') > M(g')$. Or put differently, the possibility that $M(g''') < M(g')$ is consistent with CDC.

Thus, when production complementarities across dimensions are present, an association-increasing switch in one dimension will induce further changes in achievements in other dimensions, thus requiring a comparison of deprivation matrices which involve changing deprivation in not only the switched dimension, but in other (non-switching) dimensions too. Whether this comparison should result in increasing or decreasing poverty goes beyond the scope of comparisons considered under strong rearrangement, and hence the possibility of a decrease in poverty in this case is not inconsistent with strong rearrangement or cross-dimensional convexity, nor an argument for its relaxation. Thus, in making the case for cross-dimensional convexity, it is not asserted that policy choices should ignore "efficiency" considerations on account of production complementarities between attributes, rather that incorporating them by altering the axioms of poverty measurement can cloud the ethical judgements involved in making comparisons of poverty (or social welfare).

**(v)** Finally, while the discussion above has abstracted from dimensional weights, the arguments presented above as well as the properties of the proposed measure carry over to the more general case of unequal dimensional weights. With dimensional weights specified as $w_j$ for $j = 1, \dots d$, the distribution-sensitive multidimensional poverty measures can be more generally written as:

$$M(\alpha, \beta; y) = \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{d} w_j g_{ij}^{\alpha} \right)^{\beta} \text{ for } \alpha \geq 0, \beta \geq 1 \text{ and } \sum_{j=1}^{d} w_j = 1 \qquad (12)$$

## V. Dimensional decomposition

An attractive feature of the AF method is the two-way decomposability of the aggregate poverty measures. In addition to decomposition by subgroups of population, the AF class of measures can also be decomposed by dimension (sometimes also referred to as factor decomposition[24]) such that the aggregate poverty measure can be expressed as the exact sum of contributions of individual dimensions. However, the exact dimensional decomposition presented in Alkire and Foster (2011a) is strictly a post-identification dimensional decomposition. In comparison with subgroup decomposability where one individual's contribution to the aggregate measure $M(\alpha, k; y)$ only depends on the individual's own achievement vector, one dimension's contribution to the aggregate measure also depends on the other dimensions through identification, since for a dimension to be "counted in" (or not), there must be simultaneous deprivation in at least (less than) $k - 1$ other dimensions. By contrast, in the union case identification only depends on the dimensional cut-offs, there is no cross-dimensional threshold, and all deprivations are counted in. The union-based measure $M(\alpha; y)$ thus ensures a "full" dimensional decomposition inclusive of dimensional contributions through identification.

What about dimensional decomposability of the distribution-sensitive measures? Since $\frac{\partial^2 m_i(\alpha,\beta;y)}{\partial g_{ij} \partial g_{ij'}} \neq 0$ unless $\beta = 1$, in general, the $M(\alpha, \beta; y)$ measures do not satisfy *exact* additive decomposability across dimensions.[25] This however is not a fatal blow since an exact decomposition by dimension is possible based on the Shapley value (Shorrocks, 2013). The derivation of formulae for this decomposition is discussed below.

Following the general methods set out in Shorrocks (2013), the contribution of a particular dimension $j$ can be evaluated as the expected marginal reduction in $M(\alpha, \beta; y)$ of eliminating deprivation $j$ when the expectation is taken over all possible $d!$ sequences of the elimination of the $j$-th deprivation (ranging from its being the first deprivation to be removed to its being the last). Since the expectation is formed over all possible elimination paths, these paths not only include the differing order in which the dimensions are eliminated, but, for any particular order of elimination of say deprivation $j$, also include permutation of which other one, two or $d - 1$ dimensions are eliminated prior (or subsequent) to deprivation $j$'s elimination.

With $d$ dimensions, the total number of combinations of whether a particular deprivation is present or absent is given by $2^d$ (1024 for $d = 10$, for instance, as in the UNDP's MPI). Every dimension is represented in this set of combinations as a "1" if deprivation in that dimension is present or "0" if it is absent. Let $S$ be a $2^d \times d$ matrix where each of the $2^d$ rows is a $d$-vector of 1's and 0's representing a unique combination of the presence or absence of the $d$ dimensions. For any given dimension $j$, there are then exactly half the cases ($= 2^{d-1}$) where deprivation in $j$ is present (=1) and the same number when it is absent (=0). Let these partitions of $S$ be denoted by submatrices $S_1^{(j)}$ and $S_0^{(j)}$ respectively. Thus, $S_1^{(j)} \backslash S_0^{(j)} = S$ for all $j = \{1, 2, \dots d\}$, where "$A \backslash B$" denotes the row-join operator that adds rows of $B$ below rows of $A$. Let the sum of elements of any row $t$ in $S_1^{(j)}$ be denoted $\delta(t)$ which gives the total number of present dimensions for the $t$-th combination; similarly, for any row $t'$ in $S_0^{(j)}$, let $\delta(t')$ denote the total number of present dimensions for the $t'$-th combination. The Shapley-value contribution of dimension $j$ to a multidimensional poverty measure $M$, denoted $C_j(M)$, can then be evaluated as:

$$C_j(M) = \sum_{t \in S_1^{(j)}} w(t)M(t) - \sum_{t' \in S_0^{(j)}} w(t')M(t') \tag{13}$$

where

$$w(t) = \frac{(\delta(t)-1)!\,(d-\delta(t))!}{d!} \quad \text{and} \quad w(t') = \frac{\big(\delta(t')\big)!\,(d-1-\delta(t'))!}{d!} \tag{14}$$

$M(t)$ and $M(t')$ respectively are the values of the poverty measure corresponding to $t$ and $t'$ combinations of deprivations.[26] From the results in Shorrocks (2013; equations 2.7 and 2.8), it can be shown that $\sum_{j=1}^{d} C_j(M) = M$, which establishes that the Shapley decomposition in (13) and (14) is exact. Proportional contributions of each dimension are then evaluated as $C_j(M)/M$.

The above Shapley decompositions are very general, and can be applied to virtually any multidimensional poverty measure including the distribution-sensitive measures $M(\alpha,\beta;y)$ as well as the AF measures $M(\alpha,k;y)$. The Shapley decompositions have the further advantage that they incorporate the full contribution of different dimensions to poverty measures, both through the aggregation of deprivations across dimensions as well as through the identification of the poor. These methods thus also offers a way of evaluating the *pre-identification* dimensional decomposition of counting measures.

## VI. An application to India

This section illustrates the ideas introduced in the preceding sections of the paper with an application to India using data from the Third National Family Health Survey (NFHS3) for 2005-6. In part, the motivation for using these data is that they have recently been used by Alkire and Seth (2009, 2013) for an analysis of multidimensional poverty in India following the AF methodology, and thus offer a useful point of comparison. The empirical implementation thus closely follows Alkire-Seth definitions of dimensional indicators, cut-offs and weights (as set out in Table 4.1 in Alkire and Seth, 2013).[27] In brief, Alkire-Seth use a slightly modified version of the UNDP MPI framework for India, under which there are 10 unequally-weighted indicators or dimensions, each represented by a binary variable indicating the presence or absence of the particular deprivation for a person. Amongst the 10 indicators, there are two for education with weights of 1/6th each, two for health also with weights of 1/6th each, and six indicators related to the standard of living each with a weight of 1/18th.[28] The cross-dimensional cut-off is set at 1/3; thus, a person is deemed poor if deprived in at By comparis-third of weighted dimensions. With this setup, multidimensional poverty measures $M(\alpha,\beta;y)$ and $M(\alpha,k;y)$ are constructed below for $k=1/3$ and different values of $\beta$.[29] The illustration aims to highlight five points as noted in the introduction.

*(i) A significant fraction of multidimensional poverty can be potentially missed by not counting deprivations of those below the cross-dimensional cut-off.* Table 1 reports multidimensional poverty measures both in the aggregate as well as disaggregated by the breadth of dimensional deprivation. The estimates show that about 47% of the Indian population are deprived in one-third or more of the 10 weighed dimensions and are thus deemed to be poor under the dual cut-off approach. However, there is another 41% of the population who also experience deprivations, albeit in less than one-third of weighted dimensions. If these deprivations were to be counted in, multidimensional poverty would be 29% higher, rising from 0.243 (the value of $M(\alpha,k=1/3)$) to 0.314 (the value of $M(\alpha,\beta=1)$). The deprivations of those deemed non-poor by the dual cut-off account for 22 and 10

per cent respectively of $M(\alpha, \beta = 1)$ and $M(\alpha, \beta = 2)$. If the dimensional cut-offs are considered reasonable, it seems hard to contend that more than one-fifth of all deprivations, which are assigned a weight of zero under the AF measures, represent mismeasured, accidental or chosen deprivations.

*(ii) Distribution-sensitive measures can matter for the poverty profile and subgroup contributions.* The implications of using the distribution-sensitive measures are further illustrated by considering the profile of multidimensional poverty across India's 29 states using measures with $\beta = 1, 2, 3$ incorporating increasing complementarity across multiple deprivations. The southern state of Kerala – well-known for its superior social indicators – has the lowest level of multidimensional poverty by all measures. Figure 1 expresses poverty measures for all states relative to Kerala, and shows that inter-state variation in multidimensional poverty can be seriously understated by measures that attach little or no importance to the extra burden of multiple deprivations. For instance, multidimensional poverty in Bihar is 3.5 times that in Kerala for measures with $\beta = 1$, but it is 17 times higher for measures with $\beta = 3$. Measures with no or lower orders of distribution sensitivity will tend to compress the poverty profile.

*(iii) Greater concentration of deprivations amongst the poor can* reduce *dual cut-off poverty measures.* Another respect in which the distribution-sensitive measures and the AF class of measures may evaluate certain joint distributions of deprivations differently is shown in Table 2. Starting with the actual distribution of deprivations, Table 2 considers a rearrangement of the deprivations of those deemed poor under the dual cut-off approach such that their concentration amongst the poor is maximized. Under this rearrangement, the marginal distributions for all dimensions remain unchanged. Such maximal concentration implies that many of the erstwhile poor are no longer so because post-rearrangement they are deprived in less than one-third of weighted dimensions. As a result, $H(k = 1/3)$, the incidence of AF poverty, declines from 47.4% to 33.1%. $M(\alpha, k = 1/3)$ also declines from 0.243 to 0.223. By contrast, there is only a small decline in $H(k = 0)$, the incidence of poverty under the

union approach. There is no change in $M(\alpha, \beta = 1)$ as $\beta = 1$ accords no special significance to the multiplicity of individual deprivations, but $M(\alpha, \beta = 2)$ increases sharply from 0.150 to 0.196 in response to the maximal concentration of deprivations.

*(iv) Distribution-sensitive measures offer a way of assessing how much of observed multidimensional poverty is attributable to inequality in the distribution of deprivations.* Rewriting $M(\alpha, \beta; y)$ in terms of the deprivation matrix $g^\alpha$ as $M(\alpha, \beta; g^\alpha)$, it is possible to isolate an inequality component ($\Omega$) of the distribution-sensitive measure:

$$\Omega(\alpha, \beta; g^\alpha) = M(\alpha, \beta; g^\alpha) - M(\alpha, \beta; \overline{g^\alpha}) \tag{15}$$

where $\overline{g^\alpha}$ is the deprivation matrix obtained by replacing each element $g_{ij}^\alpha$ with $g_j^\alpha$ where $g_j = \frac{1}{n}\sum_{i=1}^{n} g_{ij}$. In other words, each person's deprivation vector is replaced by the average deprivation vector. The inequality component ($\Omega$) gives a measure of how much multidimensional poverty would decline if everyone in the population had the same average deprivation vector.[30] [31]

Corresponding to the actual (observed) values of $M(\alpha, \beta; y)$, Table 3 reports the associated inequality components for different values of $\beta$. For $\beta = 1$, as $M(\alpha, 1; g^\alpha) = M(\alpha, 1; \overline{g^\alpha})$, the contribution of inequality is zero, by construction, since in the absence of complementarity, it does not matter how deprivations are distributed. However, for distribution-sensitive measures with $\beta = 2$, a little over one-third of observed multidimensional poverty is attributable to the unequal spread of deprivations across the population. This rises to nearly two-thirds as inequality is further penalized with $\beta = 3$. By contrast, the AF measures are not suited to identifying such an inequality component. For instance, for these data the inequality component for $M(\alpha, k; y)$ measure is 100% since the cross-dimensional cut-off of one-third exceeds $M(\alpha, \beta = 1)$=0.314, and hence with uniformly distributed deprivations, no one is poor.

*(v) Shapley dimensional decompositions of poverty.* Table 4 presents the Shapley dimensional decompositions of multidimensional poverty for the distribution-sensitive $M(\alpha, \beta; y)$ measures as well as the $M(\alpha, k; y)$ measures following the methods discussed in section 5. The decompositions offer a measure of the contribution of each dimension to overall poverty. The results indicate that the four dimensions of nutrition, sanitation, fuel and assets are the prime contributors to multidimensional poverty in the country accounting for about half of overall multidimensional poverty, well-above their combined dimensional weight of one-third. For these data, it turns out that the Shapley dimensional contributions for the distribution-sensitive measure $M(\alpha, \beta = 2)$ are almost identical to those for $M(\alpha, k = 1/3)$, though there are some differences with respect to those for $M(\alpha, \beta = 1)$. It also turns out that for the AF measure $M(\alpha, k = 1/3)$ itself, the post-identification dimensional contributions are not very different to the Shapley contributions which also build in contributions through identification. The similarity of dimensional decompositions across distribution-sensitive and other measures, while true of these data, is however unlikely to be a general result.

## VII. Conclusion

This paper contributes to the methodological literature on multidimensional poverty measurement. All poverty measures incorporate a set of social judgements. This is true of uni-dimensional as well as multidimensional measures of poverty, and amongst the latter, of counting measures as well as social welfare measures (invoking the distinction introduced by Atkinson, 2003). While there will always be room for some disagreement on the most appropriate set of social judgements, this paper has argued the case for distribution-sensitive multidimensional poverty measures, denoted as $M(\alpha, \beta; y)$ in the paper, which straddle the space between counting and social welfare measures, and could also be thought of as generalized counting measures. These measures, are distribution-sensitive in two ways: they satisfy the strong transfer axiom requiring regressive transfers to be poverty-increasing, and they satisfy cross-dimensional convexity (also shown to be equivalent to strong rearrangement) whereby the cumulative effect of multiple deprivations is deemed to be greater than the sum of their parts. The parameters $\alpha$ and $\beta$ calibrate these two elements of distribution-sensitivity. Indeed, by allowing $\alpha$ to vary by dimension, the distribution-sensitive measures also open up the further possibility of varying degree of interdependence across any pair of dimensions.

An empirically useful property of the counting measures of multidimensional poverty is that they are additively decomposable across subgroups of population as well as across dimensions. Dimensional decomposability has been deemed particularly attractive from a policy perspective as it offers a means of assessing how deprivations in different dimensions contribute to overall poverty, thus offering guidance on dimensional priorities for poverty reduction. But simple additive dimensional decomposition is lost when cross-dimensional convexity is introduced. The paper however demonstrates how an exact Shapley dimensional decomposition can be implemented to assess the contributions of individual dimensions. The decomposition methods described in the paper are very general in nature and can be readily applied to virtually any poverty measure, not limited to distribution-sensitive measures alone.

The paper has also sought to illustrate key properties of the distribution-sensitive measures with Indian data. Using the counting measures as a point of comparison, the application offers examples of how the use of distribution-sensitive measures can matter for assessing the magnitude of poverty, for the profile of the poor and subgroup contributions, and for the contribution of inequality in distribution of deprivations to multidimensional poverty. In many instances, such points of difference will also carry different implications for a variety of policy issues, including those related to dimensional priorities or the targeting of particular subgroups. The results for the Indian application are intended to be mainly illustrative. As in the case of uni-dimensional FGT measures relative to the widely-used headcount index, the use-value of distribution-sensitive multidimensional poverty measures will only be established through empirical practice. Additional applications will help clarify further the empirical and policy significance of distribution-sensitive measures. The fundamental appeal of these measures however ultimately rests on the properties they satisfy and how reasonable these properties are deemed to be.

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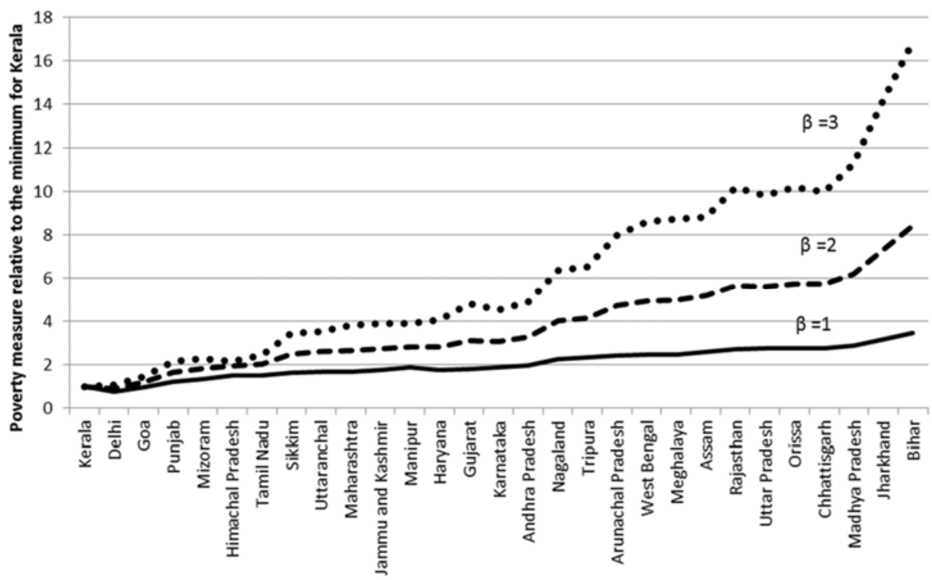
**FIGURE 1: LOWER ORDERS OF DISTRIBUTION-SENSITIVITY TEND TO COMPRESS THE MULTIDIMENSIONAL POVERTY PROFILE**




Note: The Figure shows multidimensional poverty indices for Indian States relative to Kerala (2005-6) for different degrees of distribution-sensitivity parameterized by different values of $\beta$.
Source: Author's calculations from Third National Family Health Survey data.

TABLE 1: MULTIDIMENSIONAL POVERTY MEASURES FOR INDIA, 2005-6

| **Poverty status** | **% of weighted deprivations ($c_i/d$)** | **% share in popn.** | $M(\alpha, k; y)$ ($k$=1/3) | $M(\alpha, \beta; y)$ ($\beta$=1) | $M(\alpha, \beta; y)$ ($\beta$=2) | **Average deprivation intensity** |
|---|---|---|---|---|---|---|
| Not poor by dual cut-off or union method | $c_i/d = 0$ | 11.4 | 0 | 0 | 0 | 0 |
| Not poor by dual cut-off, poor by union method | $0 < c_i/d < 1/3$ | 41.2 | 0 | 0.171 | 0.035 | 0.171 |
| Poor by dual cut-off and union method | $c_i/d >= 1/3$ | 47.4 | 0.513 | 0.513 | 0.287 | 0.513 |
| Total | | 100.0 | 0.243 | 0.314 | 0.150 | 0.314 |

| | | **Share of** | | | |
|---|---|---|---|---|---|
| **Poverty status** | **% of weighted deprivations ($c_i/d$)** | **Total depri-vations** | $M(\alpha, k; y)$ ($k$=1/3) | $M(\alpha, \beta; y)$ ($\beta$=1) | $M(\alpha, \beta; y)$ ($\beta$=2) |
| Not poor by dual cut-off or union method | $c_i/d = 0$ | 0% | 0% | 0% | 0% |
| Not poor by dual cut-off, poor by union method | $0 < c_i/d < 1/3$ | 22% | 0% | 22% | 10% |
| Poor by dual cut-off and union method | $c_i/d >= 1/3$ | 77% | 100% | 77% | 91% |
| Total | | 100% | 100% | 100% | 100% |

Note that $\alpha = 0$ for all multidimensional poverty measures reported in the Table.

Source: Author's calculations from Third National Family Health Survey data.

TABLE 2: MULTIDIMENSIONAL POVERTY MEASURES WITH ACTUAL AND MAXIMAL CONCENTRATION OF DEPRIVATIONS

| **Multidimensional headcount and poverty measure** | **Actual value** | **Value with maximum concentration of deprivations** |
|---|---|---|
| $H$ ($k$=1/3) | 47.4 | 33.1 |
| $M(\alpha, k$=1/3) | 0.243 | 0.223 |
| | | |
| $H$ ($k$=0) | 88.6 | 87.2 |
| $M(\alpha, \beta$=1) | 0.314 | 0.314 |
| $M(\alpha, \beta$=2) | 0.150 | 0.196 |

Note that $\alpha = 0$ for all multidimensional poverty measures reported in the Table.
Source: Author's calculations from Third National Family Health Survey data.

TABLE 3: INEQUALITY COMPONENT OF DISTRIBUTION-SENSITIVE MULTIDIMENSIONAL POVERTY MEASURES

| **Multidimensional poverty measure** | | $M(\alpha, \beta; y)$ | | | $M(\alpha, k; y)$ | |
|---|---|---|---|---|---|---|
| | | $\beta = 1$ | $\beta = 2$ | $\beta = 3$ | $k = 1/3$ | $k = 1/18$ |
| Actual | $M(\alpha, . ; g)$ | 0.314 | 0.150 | 0.086 | 0.243 | 0.314 |
| With uniform deprivation vector | $M(\alpha, . ; \bar{g})$ | 0.314 | 0.098 | 0.031 | 0 | 0.314 |
| Inequality component | $\Omega(\alpha, . ; g)$ | 0 | 0.052 | 0.055 | 0.243 | 0 |
| Share of inequality component | | 0% | 35% | 64% | 100% | 0% |

Note that $\alpha = 0$ for all multidimensional poverty measures reported in the Table.
Source: Author's calculations from Third National Family Health Survey data.

TABLE 4: SHAPLEY DIMENSIONAL DECOMPOSITIONS OF MULTIDIMENSIONAL POVERTY, INDIA 2005-6

| | | $M(\alpha, k{=}1/3)$ | | $M(\alpha, \beta{=}1)$ | $M(\alpha, \beta{=}2)$ |
|---|---|---|---|---|---|
| **Dimension** | **Weight (%)** | **Post-identification decomposition** | **Shapley dimensional decomposition** | **Shapley dimensional decomposition** | **Shapley dimensional decomposition** |
| **Years of schooling** | 16.7% | 11.2% | 12.3% | 9.8% | 12.0% |
| **School attendance** | 16.7% | 9.3% | 10.6% | 8.8% | 10.8% |
| **Nutrition** | 16.7% | 20.7% | 19.7% | 19.0% | 19.4% |
| **Child mortality** | 16.7% | 11.9% | 12.5% | 11.4% | 12.5% |
| **Electricity** | 5.6% | 6.0% | 5.9% | 5.8% | 6.1% |
| **Sanitation** | 5.6% | 10.6% | 10.1% | 12.3% | 10.1% |
| **Water** | 5.6% | 2.6% | 2.5% | 2.8% | 2.5% |
| **Housing** | 5.6% | 8.4% | 8.2% | 8.6% | 8.1% |
| **Fuel** | 5.6% | 11.1% | 10.5% | 12.7% | 10.5% |
| **Assets** | 5.6% | 8.2% | 8.0% | 8.8% | 8.0% |
| **Total** | 100.0% | 100.0% | 100.0% | 100.0% | 100.0% |

Note that $\alpha = 0$ for all multidimensional poverty measures reported in the Table.
Source: Author's calculations from Third National Family Health Survey data.

[1] For two recent reviews of the large literature, see Alkire et al (2015) and Aaberge and Brandolini (2015). Also see Kakwani and Silber (2008).

[2] Indeed, global MPI has been proposed as a key summary statistic to monitor progress on the SDGs (OPHI, 2014; SDSN, 2014).

[3] The term "dominance axioms" appears in Foster (2005) and relates to a subset of axioms underlying poverty measurement that are linked to stochastic dominance of distributions.

[4] The notation $variable = I(condition)$ indicates that the $variable$ equals 1 if the $condition$ is satisfied, 0 otherwise.

[5] The poverty measures $M(\alpha, k; y)$ are also a function of the $j$-vector of dimensional cut-offs, $z$; however, this argument is omitted to ease the notation.

[6] For a formal statement of these axioms, see the Supplemental Appendix.

[7] In an earlier version of the paper (Datt, 2013), this axiom was simply referred to as the transfer axiom. The adjective "strong" is used here to distinguish it from the weak transfer axiom as in Alkire and Foster (2011a), which is defined in terms of a progressive transfer involving averaging of achievement vectors of two poor persons where the post-transfer achievement matrix are obtained as the product of a bi-stochastic matrix and the pre-transfer achievement matrix, following Kolm (1977). See Supplemental Appendix for a formal definition of the weak transfer axiom.

[8] This is what Calvo and Fernandez (2012) describe as type-I errors which tend to counteract the possible upward bias due to type-II errors (incorrectly counting a non-poor individual as poor). The net effect is generally indeterminate, as also noted by Calvo and Fernandez: "Whether the dual cut-off finally mitigates or exacerbates the impact of measurement errors remains an empirical issue" (p.10). However, as they also note, the strength of type-I errors will depend in part on the proportion of population who are deprived in fewer dimensions than the cross-dimensional cut-off. Results for the Indian example in section 7 below show that this proportion can be quite large (see Table 1).

[9] For a fuller discussion of the mismeasurement and targeting issues, see Datt (2017).

[10] This is analogous to the monotonicity axiom in Sen (1976) for uni-dimensional poverty measures.

[11] This feature of the HDI or HPI is well-known and indeed has been an important motivation for the development of multidimensional poverty measures (see, for example, Alkire and Foster, 2011b).

[12] The term is due to Sen (1991) who used it to characterize this feature of utilitarianism.

[13] Levitas et al (2007) drawing upon Miliband (2006) define "deep exclusion" as "exclusion across more than one domain or dimension of disadvantage, resulting in severe negative consequences for quality of life, well-being and future life chances". Banerjee and Duflo (2011) give several examples of how interlocking deprivations in multiple dimensions can become "a mechanism for current misfortunes to turn into future poverty".

[14] This is the same as the supermodularity condition used for instance in the analysis of supermodular games, and can be interpreted as formalizing the notion of Edgeworth-Pareto complementarity (see Amir, 2005; also see Samuelson, 1974).

[15] Strictly, this measure is defined for $\alpha, \beta < \infty$. The limiting value of the measure as either parameter approaches infinity does not satisfy several of the properties discussed below. I owe this observation to a useful exchange with Suman Seth on an earlier version of the paper.

[16] This interpretation parallels the discussion of the common framework underlying the counting v social welfare approaches to multidimensional poverty measurement in Atkinson (2003).

[17] Please note that the Supplemental Appendix can also be found in the working paper version (Datt, 2017).

[18] The consideration that inequality in the distribution of deprivations should be reflected in multidimensional poverty or exclusion measures has appeared in several contributions to the literature including Jayaraj and Subramanian (2010), Rippin (2012, 2013, 2016), Dotter and Klasen (2014). For instance, the "range monotonicity" properties in Jayaraj and Subramanian (2010) and the "inequality-sensitivity" property in Rippin (2012, 2013) are motivated by considerations similar to the cross dimensional convexity axiom in this paper.

[19] By "non-additive" measures, Bourguignon and Chakravarty (2003) imply measures that are not additively decomposable across dimensions. For further discussion of dimensional decomposability, see section V.

[20] This is a simplified version of the measure in Rippin (2013, p 55) to the case where all dimensions have equal weights; the uniformity of weights has no bearing on the argument pursued here.

[21] A degree of complementarity across dimensional shortfalls is also implied by the use of cross-dimensional cut-off in the AF measures since the identification of the poor depends on simultaneous shortfalls in at least $k$ dimensions. This element of complementarity is noted in Foster (2007): "… the identification method can be

viewed as accounting for complementarities by requiring a minimum range of deprivations before it calls a person poor" (p. 15). However, there is no complementarity post-identification. The distribution-sensitive measures can thus be seen as relaxing this discontinuous treatment of complementarity across dimensional shortfalls.

[22] Such an argument has for example been pursued in Rippin (2012, 2013) where association-increasing switches are allowed to be poverty-reducing on "efficiency" grounds if there is complementarity between attributes; similar considerations also appear in Bourguignon and Chakravarty (2003), Duclos, Sahn and Younger (2006), Alkire and Foster (2011a).

[23] I am grateful to a *Review* referee and the editor for encouraging me to address this issue.

[24] The idea of factor decomposability was first stated formally in Chakravarty, Mukherjee and Ranade (1998).

[25] Strictly, one should say that the marginal effects of eliminating deprivation in each dimension do not sum up to the aggregate poverty measure. In this respect, the issue of dimensional decomposition of $M(\alpha,\beta;y)$ measures is similar to the pre-identification dimensional decomposition of AF $M(\alpha,k;y)$ measures.

[26] It is readily demonstrated that the sum of weights $w(t)$ and $w(t')$ each equals one.

[27] Estimates of the percentage of population deprived in each dimension and the corresponding AF multidimensional poverty measures in this paper are close, though not identical, to those reported in Alkire and Seth (2013); see Table A1 in Datt (2017) for a comparison. An exact replication is however not necessary for the purposes of this paper.

[28] The NFHS3 data set can be used to construct observations on the 10 indicators for a nationally-representative sample of 516,251 individuals in 108,938 households. Details on the sample design and survey methodology for NFHS3 can be found in IIPS and Macro International (2007). Further details on dimensional indicators, weights and cut-offs can be found in Datt (2017).

[29] Since we are dealing with only binary variables, this is equivalent to $\alpha = 0$.

[30] When dealing with only binary indicators, the average deprivation vector can be interpreted as the set of expected probabilities of different deprivations.

[31] Alternatively, one could also consider the contribution of inequality of deprivations *amongst the poor*, in which case the deprivation vector of each *poor* person is replaced with the average deprivation vector for the *poor*.

## Supplemental Appendix

*to the paper Datt, Gaurav (2017):*

*"Distribution-sensitive multidimensional poverty measures"*

The following gives formal definitions of various axioms referred to in the paper – other than axioms relating to strong monotonicity, strong dimensional monotonicity, cross-dimensional convexity and strong rearrangement that have already been defined in the main text. It largely draws upon the statement of these axioms in Alkire and Foster (2011a), but also refers to earlier work of Tsui (2002) and Bourguignon and Chakravarty (2003). The Appendix also gives a proof of Proposition 1 and the equivalence of cross-dimensional convexity and strong rearrangement axioms.

At the outset, it is useful to define a poverty identification function which for a given $n \times d$ matrix of achievements $y$ and a $d$-vector of dimensional cut-offs or poverty lines $z$ classifies an individual $i$ as poor or non-poor:

$$I_i(y, z) = 1 \text{ if } i \text{ is poor}; = 0 \text{ otherwise.} \tag{1}$$

This is the most general form of the poverty identification function, and the corresponding multidimensional poverty measures may be denoted $M(I; y)$ where $I$ is an $n$-vector with elements $I_i(y, z)$. For union-based multidimensional poverty measures where deprivation in at least one dimension is necessary and sufficient for an individual to be deemed poor, the poverty identification function (12) can be specialized to:

$$I_i^U(y, z) = I\big(y_{ij} < z_j \text{ for at least one } j \in \{1, \dots d\}\big) = I(c_i > 0) \tag{2}$$

where the function $I(condition)$ is an indicator function taking the value 1 if the $condition$ in parenthesis is satisfied, and 0 otherwise, and the class of union-based poverty measures could be denoted $M(I^U; y)$. The distribution-sensitive multidimensional poverty measures discussed in this paper, $M(\alpha, \beta; y)$, are thus seen to be a member of the class of union-based measures. The axioms below are stated for the general form of multidimensional poverty measures, $M(I; y)$, while the proof of Proposition 1 establishes how the axioms are satisfied by the distribution-sensitive measure, $M(\alpha, \beta; y)$.

---

**The axioms**

**Deprivation focus:** If distribution $y^*$ is obtained from $y$ by an increment in any non-deprived dimension $j'$ for individual $i'$ [i.e. there exists an $(i',j')$ such that $y_{i'j'} \geq z_{j'}$ and $y^*_{i'j'} > y_{i'j'}$ while $y^*_{ij} = y_{ij}$ for all other $(i,j) \neq (i',j')$], then $M(I;y^*) = M(I;y)$.

**Poverty focus:** If distribution $y^*$ is obtained from $y$ by an increment in any dimension $j'$ for a non-poor individual $i'$ [i.e. there exists an $(i',j')$ such that $I_{i'}(y,z) = 0$ and $y^*_{i'j'} > y_{i'j'}$ while $y^*_{ij} = y_{ij}$ for all other $(i,j) \neq (i',j')$], then $M(I;y^*) = M(I;y)$.

For union-based poverty measures, deprivation focus implies poverty focus.[31] This follows from noting that an increment in a non-deprived dimension can occur for an individual with $c_i = 0$ or $c_i > 0$. When it occurs for an individual with $c_i = 0$ such that all dimensions are non-deprived, deprivation focus implies poverty focus.

**Subgroup decomposability**: If the $n \times d$ matrix of achievements $y$ is partitioned as $y^T = [y_1^T, y_2^T]$ where $y_1$ is the $n_1 \times d$ matrix of achievements of $n_1$ individuals and $y_2$ is the $n_2 \times d$ matrix of achievements of the remaining $n_2$ individuals ($n_1 + n_2 = n$), then[31]

$$M(I;y) = \left(\frac{n_1}{n}\right) M(I;y_1) + \left(\frac{n_2}{n}\right) M(I;y_2)$$

**Replication invariance**: If distribution $y^*$ is obtained by replicating $y$ $r$-times, i.e., $y^{*T} = [y^T, y^T \ldots (\text{r times}) \ldots y^T]$, then $M(I;y^*) = M(I;y)$.

**Symmetry**: If distribution $y^*$ is obtained from $y$ as $y^* = Py$ where $P$ is a permutation matrix, then $M(I;y^*) = M(I;y)$.

**Non-triviality**: $M(I;y)$ attains at least two distinct values.

**Normalization**: $M(I;y)$ attains a minimum value of 0 and a maximum value of 1.

Note that normalization automatically ensures non-triviality.

**Weak monotonicity**: If distribution $y^*$ is obtained from $y$ by an increment in any dimension $j'$ for any individual $i'$ [i.e. there exists an $(i',j')$ such that $y^*_{i'j'} > y_{i'j'} < z_{j'}$ while $y^*_{ij} = y_{ij}$ for all other $(i,j) \neq (i',j')$], then $M(I;y^*) \leq M(I;y)$.

**Monotonicity**: If distribution $y^*$ is obtained from $y$ by an increment in any deprived dimension $j'$ for a poor individual $i'$ [i.e. there exists an $(i',j')$ such that $I_{i'}(y,z) = 1$ and $y^*_{i'j'} > y_{i'j'} < z_{j'}$ while $y^*_{ij} = y_{ij}$ for all other $(i,j) \neq (i',j')$], then $M(I;y^*) < M(I;y)$.

**Dimensional monotonicity**: If distribution $y^*$ is obtained from $y$ by an increment in any deprived dimension $j'$ for a poor individual $i'$ so that $i'$ is no longer deprived in dimension $j'$ [i.e. there exists an $(i', j')$ such that $I_{i'}(y, z) = 1$ and $y^*_{i'j'} \geq z_{j'} > y_{i'j'}$ while $y^*_{ij} = y_{ij}$ for all other $(i, j) \neq (i', j')$], then $M(I; y^*) < M(I; y)$.

**Weak transfer**: If distribution $y^*$ is obtained from $y$ as $y^* = By$ where $B$ is a bi-stochastic matrix with $b_{ii} = 1$ for all $i$ with $I_i(y, z) = 0$, then $M(I; y^*) \leq M(I; y)$.

**Strong transfer**: If distribution $y^*$ is obtained from $y$ by a transfer from $i'$ to $i''$ in a deprived dimension $j'$ such that $y_{i'j'} < z_{j'}$ and $y^*_{i'j'} = y_{i'j'} - \lambda y_{i'j'}$ and $y^*_{i''j'} = y_{i''j'} + \lambda y_{i'j'}$ for $0 < \lambda \leq 1$ , while $y^*_{ij} = y_{ij}$ for all other $(i, j) \neq \{(i'j'), (i'', j')\}$, where individual $i'$ is poorer than $i''$ [i.e. $y_{i''} D y_{i'} \Rightarrow y_{i'j} \leq y_{i''j}$ for all $j$ with strict inequality for at least one $j \in \{1, 2, \ldots d\}$], then $M(I; y^*) > M(I; y)$.

**One transfer principle**: If distribution $y^*$ is obtained from $y$ by a transfer from $i'$ to $i''$ in a deprived dimension $j'$ such that $y_{i'j'} < z_{j'}$ and $y^*_{i'j'} = y_{i'j'} - \lambda y_{i'j'}$ and $y^*_{i''j'} = y_{i''j'} + \lambda y_{i'j'}$ for $0 < \lambda \leq 1$ , while $y^*_{ij} = y_{ij}$ for all other $(i, j) \neq \{(i'j'), (i'', j')\}$, where individual $i'$ is more deprived than $i''$ in dimension $j'$ [i.e. $y_{i'j'} < y_{i''j'}$], then $M(I; y^*) \geq M(I; y)$.

**Weak rearrangement**[31]: If distribution $y^*$ is obtained from $y$ by switching between two poor individuals $i'$ and $i''$ their achievements in one or more dimension $j' \in \{1, \ldots d\}$ [i.e. $I_{i'}(y, z) = I_{i''}(y, z) = 1$, and $y^*_{i'j'} = y_{i''j'}$ and $y^*_{i''j'} = y_{i'j'}$ while $y^*_{ij} = y_{ij}$ for all other $(i, j) \neq \{(i'j'), (i'', j')\}$ ] such that while $y_{i'} D y_{i''}$ (i.e. $y_{i'}$ dominates $y_{i''}$) but that $y^*_{i'} \sim D y^*_{i''}$ (i.e. $y^*_{i'}$ does not dominate $y^*_{i''}$), then $M(I; y^*) \leq M(I; y)$.

**Proof of Proposition 1**

To begin with, corresponding to distributions $y$ and $y^*$, it is useful to define $g^{\alpha}_{ij} = \left(1 - y_{ij}/z_j\right)^{\alpha} I\left(y_{ij} < z_j\right)$ as in section 2, and $g^{*\alpha}_{ij} = \left(1 - y^*_{ij}/z_j\right)^{\alpha} I\left(y^*_{ij} < z_j\right)$, and note that in proving Proposition 1, we will be concerned with establishing specific equality or inequality relationships between $M(\alpha, \beta; y) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{d}\sum_{j=1}^{d} g^{\alpha}_{ij}\right)^{\beta}$ and $M(\alpha, \beta; y^*) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{d}\sum_{j=1}^{d} g^{*\alpha}_{ij}\right)^{\beta}$.

### *1a. Deprivation focus*.

*Given*: There is an $(i',j')$ such that $y_{i'j'} \geq z_{j'}$ and $y^*_{i'j'} > y_{i'j'}$ while $y^*_{ij} = y_{ij}$ for all other $(i,j) \neq (i',j')$.

*Proof*: For all $(i,j) \neq (i',j')$, we know that $y^*_{ij} = y_{ij}$, and thus $g^{*\alpha}_{ij} = g^{\alpha}_{ij}$. For $(i',j')$, $y^*_{i'j'} > y_{i'j'} \geq z_{j'}$, and thus $g^{*\alpha}_{i'j'} = g^{\alpha}_{i'j'} = 0$. Hence, $g^{*\alpha}_{ij} = g^{\alpha}_{ij}$ for all $(i,j)$, which establishes $M(\alpha,\beta;y^*) = M(\alpha,\beta;y)$ for $\alpha \geq 0, \beta \geq 1$. ■

### *1b. Poverty focus*.

*Given*: There exists an $(i',j')$ such that $I_{i'}(y,z) = 0$ and $y^*_{i'j'} > y_{i'j'}$ while $y^*_{ij} = y_{ij}$ for all other $(i,j) \neq (i',j')$.

*Proof*: Note that $I_{i'}(y,z) = 0$ implies $y_{i'j'} \geq z_{j'}$. Then, by proof of 1a (deprivation focus) it follows: $M(\alpha,\beta;y^*) = M(\alpha,\beta;y)$ for $\alpha \geq 0, \beta \geq 1$. ■

### *1c. Subgroup decomposability*

*Given*: $y^T = [y_1^T, y_2^T]$ where $y_1$ is the $n_1 \times d$ matrix of achievements for group 1 of $n_1$ individuals and $y_2$ is the $n_2 \times d$ matrix of achievements of group 2 of $n_2$ individuals, with the total number of individuals $n = n_1 + n_2$.

*Proof*: $M(\alpha,\beta;y) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{d}\sum_{j=1}^{d} g^{\alpha}_{ij}\right)^{\beta} = \frac{1}{n}\left[\sum_{i=1}^{n_1}\left(\frac{1}{d}\sum_{j=1}^{d} g^{\alpha}_{ij}\right)^{\beta} + \sum_{i=1}^{n_2}\left(\frac{1}{d}\sum_{j=1}^{d} g^{\alpha}_{ij}\right)^{\beta}\right]$

$= \left(\frac{n_1}{n}\right) M(\alpha,\beta;y_1) + \left(\frac{n_2}{n}\right) M(\alpha,\beta;y_2)$ for $\alpha \geq 0, \beta \geq 1$ ■

### *1d. Replication invariance*

*Proof*: This follows directly from subgroup decomposability, on noting that replication (for $r = 2$) implies $y_1^T = y_2^T$ and $n_1 = n_2 = \frac{n}{2}$. Generalizing to any $r$:

$M(\alpha,\beta;y^*) = \left(\frac{1}{r}\right) M(\alpha,\beta;y) + \cdots (\text{r times}) + \left(\frac{1}{r}\right) M(\alpha,\beta;y) = M(\alpha,\beta;y)$ for $\alpha \geq 0, \beta \geq 1$ ■

### *1e. Symmetry*

*Given*: $y^* = Py$ where $P$ is a permutation matrix.

*Proof*: Pre-multiplication by a permutation matrix only reorders (permutes) the rows of the matrix. Thus, it does not change the values of individual poverty measures, $m_i(\alpha,\beta;y_i)$, but merely reorders the individuals. Since the sum of individual poverty measures is invariant to the order of summed elements, it follows that $M(\alpha,\beta;y^*) = M(\alpha,\beta;y)$ for $\alpha \geq 0, \beta \geq 1$. ■

***1f. Normalization***

*Proof*: $M(\alpha,\beta;y)$ attains the minimum value of 0 if no one is deprived in any dimension, i.e. $g_{ij} = 0$ for all $(i,j)$. $M(\alpha,\beta;y)$ attains the maximum value of 1 if everyone is maximally deprived in every dimension, i.e. $g_{ij} = 1$ for all $(i,j)$. ■

***1g. Strong monotonicity***

*Given*: There is an $(i',j')$ such that $y^*_{i'j'} < y_{i'j'} < z_{j'}$ while $y^*_{ij} = y_{ij}$ for all other $(i,j) \neq (i',j')$.

*Proof*: For all $(i,j) \neq (i',j')$, we know that $y^*_{ij} = y_{ij}$, and thus $g^{*\alpha}_{ij} = g^{\alpha}_{ij}$. For $(i',j')$, $y^*_{i'j'} < y_{i'j'} < z_{j'}$, and thus $g^{*\alpha}_{i'j'} > g^{\alpha}_{i'j'}$ for $\alpha > 0$. Hence, for $\alpha > 0$ and $\beta \geq 1$, $m_{i'}\left(\alpha,\beta;y^*_{i'}\right) > m_{i'}(\alpha,\beta;y_{i'})$ while $m_i(\alpha,\beta;y^*_i) = m_i(\alpha,\beta;y_i)$ for all $i \neq i'$. Thus, $M(\alpha,\beta;y^*) > M(\alpha,\beta;y)$ for $\alpha > 0$ and $\beta \geq 1$. ■

***1h. Strong dimensional monotonicity***

*Given*: There exists an $(i',j')$ such that $y^*_{i'j'} \geq z_{j'} > y_{i'j'}$ while $y^*_{ij} = y_{ij}$ for all other $(i,j) \neq (i',j')$.

*Proof*: For all $(i,j) \neq (i',j')$, we know that $y^*_{ij} = y_{ij}$, and thus $g^{*\alpha}_{ij} = g^{\alpha}_{ij}$. For $(i',j')$, since $y^*_{i'j'} \geq z_{j'}$, $g^{*\alpha}_{i'j'} = 0$, while $z_{j'} > y_{i'j'}$ implies $g^{\alpha}_{i'j'} > 0$ for $\alpha \geq 0$. Hence, $g^{*\alpha}_{i'j'} < g^{\alpha}_{i'j'}$. Hence, for $\beta \geq 1$, $m_{i'}\left(\alpha,\beta;y^*_{i'}\right) < m_{i'}(\alpha,\beta;y_{i'})$ while $m_i(\alpha,\beta;y^*_i) = m_i(\alpha,\beta;y_i)$ for all $i \neq i'$. Thus, $M(\alpha,\beta;y^*) < M(\alpha,\beta;y)$ for $\alpha \geq 0, \beta \geq 1$. ■

***1i. Strong transfer***

*Given*: There exist individuals $(i',\ i'')$ where $i'$ is poorer than $i''$ [i.e. $y_{i'j} \leq y_{i''j}$ for all $j$ with strict inequality for at least one $j \in \{1, 2, \dots d\}$]. There is a transfer from $i'$ to $i''$ in a deprived dimension $j'$ such that $y_{i'j'} < z_{j'}$ and $y^*_{i'j'} = y_{i'j'} - \lambda y_{i'j'}$ and $y^*_{i''j'} = y_{i''j'} + \lambda y_{i'j'}$ for $0 < \lambda \leq 1$ , while $y^*_{ij} = y_{ij}$ for all other $(i,j) \neq \{(i'j'),(i'',j')\}$.

*Proof*: For all $j$ and $i \neq (i',i'')$, $y^*_{ij} = y_{ij}$. Thus, $m_i(\alpha,\beta;y^*_i) = m_i(\alpha,\beta;y_i)$ for all $i \neq (i',i'')$. Thus, we need to consider how $m_{i'}$ and $m_{i''}$ change as result of the transfer from $i'$ to $i''$ to determine how the aggregate poverty measure $M(\alpha,\beta;y)$ will change as a result of the transfer. Thus, from the definition of $M(\alpha,\beta;y)$, it follows that $\Delta M = M(\alpha,\beta;y^*) -$

$M(\alpha,\beta;y) \gtreqless 0$ according as $\left[(m_{i'}^{*} - m_{i'}) + (m_{i''}^{*} - m_{i''})\right] \gtreqless 0$. In particular, $n\Delta M = (m_{i'}^{*} - m_{i'}) + (m_{i''}^{*} - m_{i''})$.

Since the transfer only involves dimension $j'$, it follows that for $i = (i', i'')$ and $j \neq j'$, $y_{ij}^{*} = y_{ij}$ and thus $g_{ij}^{*\alpha} = g_{ij}^{\alpha}$. Let us denote the sums of normalized deprivations in the non-transfer dimensions as:

$q_{i'} = \sum_{j \neq j'} g_{i'j}^{*\alpha} = \sum_{j \neq j'} g_{i'j}^{\alpha}$ and $q_{i''} = \sum_{j \neq j'} g_{i''j}^{*\alpha} = \sum_{j \neq j'} g_{i''j}^{\alpha}$. Then:

$$(m_{i'}^{*} - m_{i'}) = d^{-\beta}\left[\left(g_{i'j'}^{*\alpha} + q_{i'}\right)^{\beta} - \left(g_{i'j'}^{\alpha} + q_{i'}\right)^{\beta}\right] > 0 \text{ because } g_{i'j'}^{*\alpha} > g_{i'j'}^{\alpha}$$

$$(m_{i''}^{*} - m_{i''}) = d^{-\beta}\left[\left(g_{i''j'}^{*\alpha} + q_{i''}\right)^{\beta} - \left(g_{i''j'}^{\alpha} + q_{i''}\right)^{\beta}\right] \leq 0 \text{ because } g_{i''j'}^{*\alpha} \leq g_{i''j'}^{\alpha}$$

We now prove the proposition in three parts.

(i) Consider first the case where $\beta = 1$. In this case:

$(m_{i'}^{*} - m_{i'}) = d^{-1}\left[g_{i'j'}^{*\alpha} - g_{i'j'}^{\alpha}\right]$ and $(m_{i''}^{*} - m_{i''}) = d^{-1}\left[g_{i''j'}^{*\alpha} - g_{i''j'}^{\alpha}\right]$

Thus: $n\Delta M = (m_{i'}^{*} - m_{i'}) + (m_{i''}^{*} - m_{i''}) = d^{-1}\left[\left(g_{i'j'}^{*\alpha} - g_{i'j'}^{\alpha}\right) - \left(g_{i''j'}^{\alpha} - g_{i''j'}^{*\alpha}\right)\right]$

Two cases can be distinguished: (a) individual $i''$ remains deprived in dimension $j$ even after receiving the transfer from individual $i'$, or (b) $i''$ is not deprived in dimension $j$ post-transfer. In case (a), since $g_{i'j'}^{\alpha} \geq g_{i''j'}^{\alpha}$ and for $\alpha > 1$ normalized deprivations $g_{ij}^{\alpha}$ are strictly convex in $g_{ij}$, it follows that $n\Delta M > 0$ for $\alpha > 1$. The value of $n\Delta M$ is even higher in case (b) as the value of $\left(g_{i''j'}^{\alpha} - g_{i''j'}^{*\alpha}\right)$ is lower since $g_{i''j'}^{*\alpha} = 0$ in this case. Thus, $n\Delta M > 0$ for $\alpha > 1$ in both cases (a) and (b). This proves that $M(\alpha,\beta;y)$ satisfies strong transfer axiom for $\alpha > 1$ and $\beta = 1$.

(ii) Now, consider the case where $\alpha = 1$. In this case:

$$n\Delta M = d^{-\beta}\left[\left\{\left(g_{i'j'}^{*} + q_{i'}\right)^{\beta} - \left(g_{i'j'} + q_{i'}\right)^{\beta}\right\} - \left\{\left(g_{i''j'} + q_{i''}\right)^{\beta} - \left(g_{i''j'}^{*} + q_{i''}\right)^{\beta}\right\}\right]$$

Since $(g_{i'j'} + q_{i'}) \geq (g_{i''j'} + q_{i''})$ and since for $\beta > 1$, by cross-dimensional convexity (see ***1j*** below), the individual poverty measures $m_i = d^{-\beta}\left(g_{ij} + q_i\right)^{\beta}$ are strictly convex in $g_{ij}$, it follows by the same logic as in case (i) that $n\Delta M > 0$ for $\beta > 1$, thus proving that $M(\alpha,\beta;y)$ satisfies strong transfer axiom for $\alpha = 1$ and $\beta > 1$.

(iii) Now, consider the case where both $\alpha > 1$ and $\beta > 1$. Both $\alpha, \beta > 1$ doubly ensures that individual poverty measures $m_i = d^{-\beta}\left(g_{ij}^{\alpha} + q_i\right)^{\beta}$ are strictly convex in $g_{ij}$. On further noting that $m_{i'} \geq m_{i''}$ proves that $n\Delta M > 0$ and $M(\alpha, \beta; y)$ satisfies strong transfer axiom for $\alpha > 1$ and $\beta > 1$.

Cases (i), (ii) and (iii) together establish that $M(\alpha, \beta; y)$ satisfies strong transfer axiom for $\alpha > 1, \beta \geq 1$ or $\alpha \geq 1, \beta > 1$. ■

### *1j. Cross-dimensional convexity*

*Proof*: Note that $M(\alpha, \beta; y)$ can be written as an average of individual poverty measures, $m_i(\alpha, \beta; y_i)$:

$$M(\alpha, \beta; y) = \frac{1}{n}\sum_{i=1}^{n} m_i\,(\alpha, \beta; y_i) \quad \text{where } m_i(\alpha, \beta; y_i) = \left(\frac{1}{d}\sum_{j=1}^{d} g_{ij}^{\alpha}\right)^{\beta} \qquad (3)$$

and

$$\frac{\partial^2 m_i(\alpha,\beta;y)}{\partial g_{ij}\partial g_{ij'}} = \frac{\alpha^2\beta(\beta-1)g_{ij}^{\alpha-1}g_{ij'}^{\alpha-1}}{d^2} m_i(\alpha, \beta - 2; y_i) \;\; > 0 \quad \text{for } \beta > 1 \qquad (4)$$

That establishes $M(\alpha, \beta; y)$ satisfies cross-dimensional convexity for $\alpha \geq 0, \beta > 1$ ■

### *1k. Strong rearrangement*

*Given*: $y_{i'}Dy_{i''}$, both $i'$ and $i''$ are poor (i.e. $c_{i''} \geq c_{i'} > 0$), and $y_{i'j'}^{*} = y_{i''j'}$ and $y_{i''j'}^{*} = y_{i'j'}$ while $y_{ij}^{*} = y_{ij}$ for all other $(i, j) \neq \{(i'j'), (i'', j')\}$ and $y_{i'}^{*} \sim Dy_{i''}^{*}$.

*Proof*: This is proved in two parts: (i) $M(\alpha, \beta; y^*) = M(\alpha, \beta; y)$ if $j'$ is not a deprived dimension for $i''$ in $y$ or if $y_{i'j'} = y_{i''j'}$, and (ii) $M(\alpha, \beta; y^*) < M(\alpha, \beta; y)$ otherwise.

(i) First consider If $j'$ is not a deprived dimension for $i''$ in $y$. Given $y_{i'}Dy_{i''}$, it thus follows that $j'$ is also not a deprived dimension for $i'$ in $y$. Then, by the deprivation focus axiom, $M(\alpha, \beta; y^*) = M(\alpha, \beta; y)$.

Next consider, $y_{i'j'} = y_{i''j'}$. With the switch, then $y_{i'j'}^{*} = y_{i'j'}$ and $y_{i''j'}^{*} = y_{i''j'}$, thus implying $m_i(\alpha, \beta; y_i^*) = m_i(\alpha, \beta; y_i)$ for all $i = (i', i'')$, which in turn implies that $M(\alpha, \beta; y^*) = M(\alpha, \beta; y)$.

(ii) Now consider the case where $j'$ <u>is</u> a deprived dimension for $i''$ in $y$ and $y_{i'j'} \neq y_{i''j'}$. Given $y_{i'j'} \neq y_{i''j'}$, the condition $y_{i'}Dy_{i''}$, must imply that $y_{i'j'} > y_{i''j'}$. Since $j'$ <u>is</u> a deprived dimension for $i''$, it then follows that $g_{i''j'}^{\alpha} > g_{i'j'}^{\alpha}$. A switch in dimension $j'$ for individuals $i'$ and $i''$ further implies that $g_{i'j'}^{*\alpha} = g_{i''j'}^{\alpha}$ and $g_{i''j'}^{*\alpha} = g_{i'j'}^{\alpha}$. Following the development as in

***1i***(ii) above for the strong transfer axiom, the (additive inverse of the) change in multidimensional poverty following the switch can be written as:

$$-nd^{\beta}\Delta M = \left\{\left(g^{\alpha}_{i''j'} + q_{i''}\right)^{\beta} - \left(g^{\alpha}_{i''j'} + q_{i'}\right)^{\beta}\right\} - \left\{\left(g^{\alpha}_{i'j'} + q_{i''}\right)^{\beta} - \left(g^{\alpha}_{i'j'} + q_{i'}\right)^{\beta}\right\}$$

Upon noting that $q_{i''} - q_{i'} \geq 0$ and $g^{\alpha}_{i''j'} > g^{\alpha}_{i'j'}$ , it follows that the right-hand-side of the above equation is positive ($> 0$) for $\beta > 1$, thus establishing that in this case

$M(\alpha,\beta;y^{*}) < M(\alpha,\beta;y)$ for $\beta > 1$. ■

**Proof of the equivalence of cross-dimensional convexity and strong rearrangement**

In considering a switch of achievements between individuals $i'$ and $i''$ as per the strong rearrangement axiom, we need to compare multidimensional poverty associated with distributions $y$ and $y^{*}$ before and after the switch. As the switch involves only the two individuals, given subgroup decomposability, this comes down to comparing individual poverty of $i'$ and $i''$ before and after the switch.

Let the pre-switch deprivation vectors be given by

$g_{i'} = \{g_{i'1}, g_{i'2}, \dots g_{i'd}\}$ and $g_{i''} = \{g_{i''1}, g_{i''2}, \dots g_{i''d}\}$

Since prior to switch $y_{i'}Dy_{i''}$, let $g_{i'j} \leq g_{i''j}$ for all $j$ and without loss of generality, for say dimension $d$, $g_{i'd} < g_{i''d}$, or equivalently $g_{i'd} + \delta = g_{i''d}$ for $\delta > 0$. Then, writing individual poverty levels of $i'$ and $i''$ as functions of their deprivation vectors, the change in aggregate poverty after the switch in dimension $d$ can be written

$$\begin{aligned}\Delta M = M(..;y^{*}) - M(..;y) = {} & [m(g_{i'1}, g_{i'2}, \dots g_{i'd} + \delta) - m(g_{i'1}, g_{i'2}, \dots g_{i'd})] - \\ & [m(g_{i''1}, g_{i''2}, \dots g_{i''d}) - m(g_{i''1}, g_{i''2}, \dots g_{i''d} - \delta)]\end{aligned}$$

Note that both terms in square parentheses involve an increment in deprivation $d$ by $\delta > 0$. We are now in a position to prove the equivalence.

Given cross-dimensional convexity, the increment in poverty in the second square bracket with respect to individual $i''$ is greater than that in the first square bracket with respect to individual $i'$ because $g_{i''j} \leq g_{i''j}$ and $g_{i''d} < g_{i'd}$. Since $i''$ is more deprived in one or more dimensions relative to $i'$, the same increment in deprivation $d$, given cross-dimensional convexity, occasions a greater increment in poverty for $i''$. Thus, cross-dimensional poverty implies $\Delta M < 0$, which implies strong rearrangement.

Conversely, given strong rearrangement, $\Delta M < 0$ (by definition of strong rearrangement). But this in turn implies that the same increment in deprivation $d$ occasions a greater increment in poverty for $i''$ who is more deprived in one or more dimensions relative to $i'$. This in turn ensures cross-dimensional convexity, which completes the proof. ■