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MULTI-LEVEL PROGRAMMING

by

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*Wilfred Candler and Roger Norton**

1. Introduction

The subject of this paper is the following problem: the behavior of a system nay be adequately described by a mathematical programming model, but there may be external conditional controls imposed on the system. If the controls are conditional in the sense that their values depend upon the system's reactions to them, then they cannot be represented by exogenous constraints on the mathematical programming model. To solve the problem, an algorithm is required which permits the simultaneous and interdependent functioning of two optimization processes.

This problem is relevant to biology, engineering, and other disciplines, but economics is the context here. The problem already has been confronted in various guises in economics, as indicated below.

Mathematical programming models of economic systems are frequently used to represent maximization of policy objectives, subject to technical constraints (such as input-output accounts) and a limited number of

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behavioral constraints (such as a consumption function).[1/] Policy instruments to achieve the objectives are included with widely varying degrees of specificity.

More recently, programming models have been used to simulate the response of decentralized behavior to policies rather than to maximize a policy function directly. In this case, a market structure is specified (competitive, monopolistic) and individual decision rules may also be stated (profit maximization, risk aversion, etc.), and the optimization procedure is employed as a device to ensure that the model's solutions reflect that behavior. This use of mathematical programming was first suggested by Samuelson [16], and later it was implemented in models for agriculture [5, 18].

In the first case, a policy objective function is maximized, and in the second case a behavioral objective function is maximized. When the latter is done, the "policy space" is searched informally at a discrete number of points by conducting solutions under alternative exogenous values of the policy instruments. In neither case, however, is the policy problem completely formulated. A full statement of the policy problem recognizes explicitly two sub-problems: a) the behavioral simulation subproblem, or the question of forecasting the reactions to policies of decentralized decision makers, and b) the policy optimization subproblem itself, which is the question of choosing according to a set of policy preferences. In other terms, they are the positive and normative subproblems, respectively.

1/ At the economy-wide level, some early and well-known examples are the models of Sandee [17] and Manne [14].

In some situations, especially in agriculture, the behavioral simulation subproblem is most adequately formulated in terms of maximization subject to inequality constraints, or mathematical programming. However, when this is so, replacing the behavioral objective function by a policy objective function would destroy the behavioral formulation. The model of reactions to policies would no longer have a clear conceptual or empirical basis, and to that extent the policy results from the model would be less meaningful. 2/

This paper offers a procedure for combining the policy subproblem and the behavioral simulation subproblem in a programming model which contains two distinct and operative objective functions. When formulated in this way, the procedure is seen to be a generalization of mathematical programming. As mathematical programming algorithms cannot solve this problem, a new algorithm has been developed. It is summarized in section 4 below, and a numerical example is reported in section 5.

Previous treatments of the problem of two objective functions, in the context of the multi-level planning literature, all have been concerned with specifications which could in principle be represented as a single

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2/ The same argument holds for imposing on the behavioral simulation model a set of policy-motivated constraints which have neither behavioral nor technical interpretation. The model's meaning is not clear when a behavioral objective function is maximized subject to say, a minimum employment constraint. (How is the constraint enforced in reality?) The meaning is clear, however, if the behavioral maximization takes place subject to, say, a wage subsidy which is designed to promote employment.

(large-scale) mathematical programming problem [9, p.210].[3] For various reasons, including lack of full information at the outset, solution procedures have been sought through a sequence of mathematical programs [10, 13].[4]

Other proposed methods, such as "goal programming", in effect address the problem of choosing weights in the policy objective [12], but they suppress the two-way interaction between the objective functions.[5]

The notion of two interdependent groups of economic actors has been expressed by Theil, in the context of econometric estimates of behavior [19, pp. 372-75]. If the behavioral structure has been maintained over time, and if there has been adequate experience over the relevant

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3/ In this respect, of course, the previous problem specifications differ from the one given in this paper. The multi-level planning methods use the iterations as a device for successively tracing out points in the feasible space of the behavioral subproblem; however, the control over both behavioral and policy variables is retained at the level of the policy subproblem. By contrast, in multi-level programming, as defined in this paper, control over behavioral variables, in reaction to any policy option, is left at the (decentralized) behavioral level.

4/ In any event, the iterative multi-level planning procedures are so cumbersome that in practice an acceptable rate of numerical convergence is rarely attained. In fact, guaranteed convergence of the sequence of iterations has been proven only for particular statements of the objective function and only for the infinite-iteration case [8, 13]. In recognition of these practical difficulties Kornai [9] has suggested abandoning convergence as an aim, arguing that in practice only a few iterations would be carried out if the two objective functions were specified in a realistic way. (For an example of numerical multi-level planning procedures, see [6].)

5/ "Reactive programming" [20], while recognizing two groups of economic actors, is an iterative procedure for attaining a solution to just the behavioral simulation problem.

range of policy variables, then the econometric approach can yield very satisfactory estimates of the optimal values of the policy variables. Also in the context of econometrics, Marschak gave a very clear definition of the multi--level programming problem in a classic early work [15, pp.1-2].

The multi-level programming problem also is closely related to game theory. It is shown below (section 4) that, in terms of problem formulation, the Stackelberg game is a special case of multi-level programming. However, there is a very important difference in that solution of a game requires explicit knowledge of "reaction functions," whereas they are allowed to be implicit in the activity-analysis format of multi-level programming.

In the following sections of this paper, multi-level programming is explicitly defined, its relation to conventional mathematical programming is given, an algorithm is offered for its solution when there are only two objective functions and when all functions are linear, and a small numerical example is presented. In conclusion, some remarks are made about uniqueness of solutions. The algorithm is a modified version of the simplex algorithm. The numerical example is adapted from Louwes, Boot, and Wage's analysis of the optimal pricing of milk products in the Netherlands [11].

2. *Multi-level Programming: The Two-Level Case*

The essence of the multi-level programming problem is that policy makers are attempting to maximize a policy objective function, while controlling only a sub-set of the variables. The variables controlled by policy makers are called policy variables, the variables which enter the policy makers' objective function are called impact variables, while the variables which are controlled by other decision makers are referred to as behavioral variables. All three classes of variables are endogenous. In the two-level case, the multi-level programming problem may be stated as follows.

Find a vector $\underline{x} = (\underline{x}_1, \underline{x}_2)$ such that:

$$f_2 = \max_{\underline{x}_2} (\underline{c}_2' \underline{x}) \tag{1}$$

subject to

$$f_1 = \max_{\underline{x}_1 | \underline{x}_2} (\underline{c}_1' \underline{x}) \tag{2}$$

and

$$A \underline{x} \leq b \tag{3}$$

$$x \geq 0 \tag{4}$$

where $\underline{x}_1$ is a vector of behavioral and impact variables,

$\underline{x}_2$ is a vector of policy variables, and

A, $\underline{b}$, $\underline{c}_1$, and $\underline{c}_2$ are appropriate matrices and vectors of constants.

For a given level of $\underline{x}_2$, (2), (3), and (4) define a linear programming problem. However, (1) through (4) is not a linear programming problem; hence the need for a new name and a new algorithm. Extensions of the proposed algorithm if (1) or (2) is quadratic, or if $\underline{x}$ is mixed integer, should not represent a major problem.

As for linear programming, any non-negative vector $\underline{x}$ is called a solution. Any solution satisfying (3) and (4) is called a (primal) feasible solution. Any feasible solution maximizing (2) for given $\underline{x}_2$ is called a behavioral optimal solution. Any behavioral optimal solution, maximizing (1) is called a policy optimal solution.

This formulation of the problem may be compared readily with Theil's approach in the reference cited above [19]. The nomenclature comparisons are as follows:

	Theil	Multi-level Programming
1.	instrument or control variable	policy variable
2.	non-control variable	behavioral variable
3.	exogenous variable	[reflected in model structure, especially right-hand side values]
4.	[no specific name but quoted in his example]	impact variable

Interestingly, Theil refers to values of the non-control variables as being estimated by "an econometric model of the 'behavioral' aspect".

3. *Typical Problem Structures*

Amplifying the notation slightly by introducing the partitioning $\underline{x}_1 = (\underline{x}_o, \underline{x}_1)$, a typical multi-level programming problem would look as follows:

$$f_2 = \max_{\underline{x}_2} (\underline{c}_2' \underline{x}_o) \tag{5}$$

subject to:

$$f_1 = \max_{\underline{x}_1^* | \underline{x}_2} (\underline{c}_1 \underline{x}_1) \tag{6}$$

$$A_{11} \underline{x}_1^* + A_{12} \underline{x}_2 \leq \underline{b} \tag{7}$$

$$-I\underline{x}_o + A_{21} \underline{x}_1^* + A_{22} \underline{x}_2 = 0 \tag{8}$$

$$\underline{x} \geq 0 \tag{9}$$

where now $\underline{x}_o$ is a vector of impact variables,

$\underline{x}_1^*$ is a vector of behavioral variables, and

$\underline{x}_2$ is a vector of policy variables

This partitioning is meant to represent a fairly common situation in which:

(i) Only the impact variables $\underline{x}_o$ affect the policy makers' objective function;

(ii) Only the behavioral choice variables $\underline{x}_1^*$ and (possibly some of) the impact variables $\underline{x}_o$ affect the behavioral objective function;

(iii) A_{11} is a technological matrix of resource requirements;

(iv) The matrix A_{12} expresses the effect of the policy variables $\underline{x}_2$ on resource availability (a policy which increases resource availability, such as investment in new irrigation supplies, is represe ted by a negative element of A_{12});

(v) The vector $\underline{b}$ represents the level of resource availability prior to policy intervention;

(vi) A_{21} is a matrix of the effects of the behavioral variables $\underline{x}_1^*$ on the impact variables $\underline{x}_o$; and

(vii) A_{22} is a matrix of the direct effects of the policy variables $\underline{x}_2$ on the impact variables $\underline{x}_o$ (in many cases this matrix would be zero and so policies would have to achieve their impacts indirectly, viz., through the matrices A_{12} and A_{21}).

4. *Mathematical Programming as a Special Case*

Mathematical Programming, P1

A mathematical programming problem: P1 may be written in general as:

Find $\underline{x}$ such that

$$f(\underline{x}) \rightarrow \max$$

subject to: $g_i(\underline{x}) = 0, \quad i = 1, \ldots, m.$

Multi-Level Programming, P2

A multi-level programming problem: P2 may he written in general as:

Find $\underline{x}_j$, $j = 1, \ldots, n$, such that

$$f_j(\underline{x}_j \mid \underline{x}_k , \quad k = j + 1, \ldots , n) \rightarrow \max, \; j = 1, \ldots, n$$

subject to:

$$g_i(\underline{x}_j \, , \; j = 1, \ldots , n) = 0, \quad i = 1, \ldots , m.$$

Mathematical programming (P1), can be seen as a special case of Multi-level programming (P2), since if n=1, then P2 becomes:

Find $\underline{x}_1$ such that

$$f_1(\underline{x}_1) \rightarrow \max$$

subject to:

$$g_i(\underline{x}_1) = 0, \quad i = 1, \ldots , m,$$

which is P1, mathematical programming.

If n = 2, then P2 becomes the following problem:

Find $\underline{x}_1$, $\underline{x}_2$ such that

$$f_2(\underline{x}_2) \rightarrow \max$$

subject to: $$f_1(\underline{x}_1 \mid \underline{x}_2) \rightarrow \max$$

and $$g_i(\underline{x}_1 , \underline{x}_2) = 0 \quad i = 1, \ldots , m.$$

This last problem is multi-level programming in the two-level case, as spelled out in section 2 above.[6/] The problem P2 is multi-level programming in general.

6/ In this statement of the problem, it is important to recognize that the policy choice variables ("control variables") which are manipulated to maximize f_2 are not identical with the set of impact variables ("target variables" or "state variables") $\underline{x}_2$. If they were identical, then the problem collapses to mathematical programming, because then the second objective function f_1 would have no influence on the outcome. The domain over which maximization occurs is made clear in the problem descriptions in section 3 and 4 above, so this qualifier is not needed there.

The Stackelberg game is another special case when n=2 and when the objective function f_2 represents the behavior of the "leader" and f_1 represents the behavior of the "follower". As noted above, however, multi-level programming deals with cases in which the reaction functions of game theory are not known and hence game-theoretic solution procedures are inapplicable. Nevertheless, insofar as multi-level programming is a relevant formulation for numerical analysis of macro-economic policy problems, then it can be seen that the leader-follower version of the Stackelberg game is an apt conceptual analogy to the policy problem.

The usefulness of multi-ltvel programming depends very much on having an algorithm which will guarantee numerical solutions,, so the next section is devoted to development of such an algorithm. No doubt others could develop more efficient algorithms; the concern here is to show that at least one solution procedure exists and to prove that it converges.

5. *Optimality Conditions for the Algorithm*

The suggested algorithm is developed in the context of an updated simplex tableau, where all functions are assumed linear, continuous and convex. That is, the algorithm revolves around the question of which variables to bring into the basis at each iteration. The linear case is addressed, and the notation of section 2 is adopted. For simplicity the vectors $\underline{c}_2'$ and $\underline{c}_1'$ are stacked on the matrix A to give:

$$T = \begin{bmatrix} \underline{c}_2 \\ \underline{c}_1 \\ A \end{bmatrix} \tag{10}$$

The initial simplex tableau can be written

$$T\underline{x} = \begin{bmatrix} 0 \\ 0 \\ \underline{b} \end{bmatrix} \quad (11)$$

For the kth iteration, $\underline{x}$ can be partitioned into $\underline{x}_b$ and $\underline{x}_n$, where the basic variables appear in $\underline{x}_b$ and the other, non-basic, variables appear in $\underline{x}_n$.

Making a corresponding partition of T into T_b and T_n allows us to express the basic variables in terms of the non-basic variables:

$$\underline{x}_b = T_b^{-1}\underline{b} - T_b^{-1}T_n\underline{x}_n \quad (12)$$

It is the elements on the right of (12) which we refer to as the updated tableau at the kth iteration. It is convenient to represent these elements schematically as in Table 1.

It is also convenient, for algebraic purposes to refer to the elements in Table 1 as t_{ij}, where j=1 for right-hand side elements, j=2, ... , e for non-policy, non-basic variables[7/] and j = e+1, ..., n for non-basic policy variables. Also, i=1 for the policy objective and i=2 for the behavioral objective, and other subscripts have interpretations assigned to them in the text.

7/ The label "non-policy" variables embraces behavioral, impact, and disposal (slack) variables.

Table 1: Schematic Representation of kth Iteration Tableau

Basic activities	RHS	Non-Basic Activities: Non-Policy	Non-Basic Activities: Policy
Policy objective	+	+ - + -	+ - + -
Behavioral objective	+	+ + - -	+ + - -
Non-policy variables	+ +	+ + - +	+ - - -
Policy variables	+ +	- + - -	+ + + -

The updated tableau is represented in "standard" form in Table 1. The first two rows are taken to refer to the updated coefficients in the policy and behavioral objective functions respectively. Then come two sets of basic rows for non-policy and policy variables respectively. The "right hand side" of mathematical programming appears on the left, followed by two sets of non-basic activities for non-policy and policy variables respectively. The RHS signs are meant to indicate that once feasibility has been attained, the basic behavioral variables will be non-negative. Also, in general, the coefficients elsewhere in the tableau may be zero or may take positive or negative signs. It is assumed throughout that where an activity is selected to enter the basis, a simplex iteration is carried out in such a way as to maintain the (primal) feasibility of the basic behavioral, disposal and policy variables. Once primal feasibility is attained, it is maintained.

The modelling convention used means that bringing in an activity with a negative coefficient increases the right hand side value for the corresponding variable. (In particular, the behavioral objective function f_1 is at a maximum, for given values of the policy variables, when all non-policy coefficients in the second row of the table are non-negative).

Degeneracy problems are assumed handled by the addition of "small" disturbances, which uniquely resolve any ties [3], without affecting the significant figures in the solution.

Conditions for Optimality

Before developing the algorithm, it is useful to develop sufficient conditions optimality. Given a basic feasible solution in standard form, then [4, Theorem 2]:

Sufficient Condition for Behavioral Optimality:

$$t_{2j} \geq 0 \qquad j = 2, \ldots, e$$

This simply says that it is not possible to increase the value of the behavioral objective function by bringing into the basis any non-basic non-policy variables $x_2, \ldots, x_e = 0$.

Two necessary conditions for policy optimality can be stated as follows:

Necessary Condition 1: That the solution be a behavioral optimal solution.

Necessary Condition 2: $t_{1j} \geq 0 \quad j = e+1, \ldots, n$

This second condition follows from the same theorem of Dantzig [4, Theorem 2], and states that for non-basic policy variables $x_{e+1}, \ldots, x_n = 0$, it is not possible to improve the value of the policy objective function by increasing the level of one of those variables.

A third necessary condition relates to changing the levels of the basic policy variables when a new variable enters the basis. Let us suppose that the basic policy variables are found in the last rows of the tableau; so that their current values are given by t_{i1}, $i = q, q+1, \ldots, m$. Then the effect on the levels of the basic policy variables of introducing activity j into the basis is measured by t_{ij}, $i = q, q+1, \ldots, m$. The effect of this change on the level of the policy objective is measured by t_{1j}, while the effect on the behavioral objective is measured by t_{2j}. (The possibility of increasing the policy objectfve by changing the level of non-basic policy variables, has already been covered by Necessary Condition 2.)

For any small change, $t_{q,p}, t_{q+1,p}, \ldots, t_{m,p}$, in the level of the basic policy variables (induced by the introduction of a non-basic activity p), the non-basic non-policy variable, j_p, which will be brought into the basis for a behavioral optimal solution is identified by the condition:

$$j_p = \left[j \;\middle|\; \min_j \left(\frac{t_{2j}}{\sum_{i=q}^{m} t_{ip} t_{ij}} \;\middle|\; \sum_{i=q}^{m} t_{ip} t_{ij} > 0, \quad j = 2, \ldots, e \right) \right] \tag{13}$$

where for a policy optimal solution we know that $t_{2j} \geq 0$, from Necessary Condition 1. This allows us to state the third necessary condition:

Necessary Condition 3: Given Necessary Conditions 1 and 2,

$$t_{1j_p} \geq 0 \quad p = 2, \ldots, e$$

where j_p is defined in (13)

Proof: Any change in the solution $t^k_{11}, \ldots, t^k_{1m}$ can be expressed as a linear combination of the non-basic variables $x_2, \ldots, x_n$. If Necessary Condition 2 holds, the net contribution of $x_{e+1}, \ldots, x_n$ to the policy objective will be non-positive. Hence, we only require that the contribution of variables $x_2, \ldots, x_e$ also be non-positive. But if Necessary Condition 3 holds for all j_p satisfying (13), then the contribution of activities $x_2 \ldots x_e$. will also be non-negative.

Sufficient Condition for Local Optimum

Necessary conditions 1, 2 and 3 taken together- are sufficient for a local optimum to the policy problem.

Sufficient Condition for Global Optimum

The convexity assumption ensures that any local optimum to the policy problem is a global optimum.

6. *Fixed and Free Policy Variables*

Fixing the Levels of the Policy Variables

The algorithm calls for the policy variables to be temporarily fixed in values from time to time. In terms of Table 1, it is only necessary to store the (positive) levels of the basic policy variables elsewhere and replace the corresponding right hand side entries by zeros.

This is represented schematically in Table 2, where the "storage location" is the section below the double line.

Table 2: Tableau with Policy Variables Fixed

Basic activities	RHS	Non-basic activities: Non-policy	Non-basic activities: Policy — Free	Non-basic activities: Policy — Fixed
Policy objective	+	+ − + −	+ − + −	0 0
Behavioral objective	+	+ + − −	+ + − −	0 0
Non-policy variables	+	+ +	+ −	0 0
	+	− +	− −	0 0
Policy variables	0	− +	+ +	1 0
	0	− −	+ −	0 1
Fixed policy variables			0 0 0 0	+ +

To keep the policy variables at the levels indicated in Table 2, it is only necessary for any non-policy variables entering the basis to pivot on the policy rows, if possible, regardless of the sign of the pivot. This preferential pivoting is an instruction in this phase of the algorithm. Bringing in non-policy variables (with negative entries in the behavioral objective function) it is possible to maximize the behavioral obfective, as shown in Table 3. There may, or may not, be policy variables remaining in the basis. In this phase, certainly any free policy variables that remain in the basis will be at zero level. Table 3 is referred to as behavioral optimal, since it maximizes the value of the behavioral objective function, given the levels of the fixed policy variables.

Table 3: A "Behavioral Optimal" Tableau

Basic activities	RHS	Non-basic activities									
		Non-policy				Policy					
						Free				Fixed	
Policy objective	+	+	−	+	−	+	−	+	−	+	0
Behavioral objective	+	+	+	+	+	+	+	−	−	+	0
Son-policy variables	+	+		+		+		=		+	0
	+		−		+		−	+		−	0
	+	−		+		+					
Policy variables	0		+		−		+		−	+	0
Fixed policy variables								0	0	+	+
						0	0				

Freeing the Level of a Policy Variable

Since the algorithm calls for the fixing, or removal, of policy variables from the basis, provision also needs to be made for the "freeing", or re-entry of fixed variables into the basis. In Table 3 there are two fixed policy variables, represented by the two columns on the extreme right. One is in the basis at level zero, and the other is non-basic. To "free" the basic policy variable it is only necessary to replace the zero in the "right hand side" with the fixed value of the variable.

For the non-basic fixed variable, things are a little more complicated since we may wish to increase or decrease the value of this variable. To increase its value, we pivot on an appropriate positive pivot, and when the variable is basic add the fixed value to the right-hand side in the row for that variable.

To decrease a fixed policy variable x_j we estimate:

$$\theta_j = \min_i \left(-t_{i1} / t_{ij} \;\middle|\; t_{ij} < 0\right)$$

where t_{ij} is the element in the ith row and jth column of the currently updated tableau. If θ_j is less than or equal to the fixed value, we pivot on the t_{ij} used to define θ_j, and then add the fixed value to the right hand side, once activity j is basic.

If θ_j is greater than the fixed value, $\bar{t}_j$, we simply change the right hand side to t'_{i1} where:

$$t'_{i1} = t_{i1} - \bar{t}_j \, t_{ij}, \qquad i = 1, \ldots, m$$

and reset the fixed value $\bar{t}_j$ to zero.

4 . *The Algorithm*

Overall, the algorithm consists of five steps which can be understood in the context of simplex operations and the foregoing discussion. The five steps are as follows:

1. Set up the problem in simplex tableau form. Find a feasible solution using any linear programming algorithm. Go to Step 2.
2. Fix the level of the policy variables and find the corresponding behavioral optimal solution. Go to Step 3.

3. Free the levels of the policy variables, and bring into (or eliminate from) the basis any policy variables for which the entries in both the objective functions are negative (or positive). If no change of basis is required, go to Step 4; otherwise return to Step 2.

4. Optimize with respect to the policy variables and policy objective function. If no change of basis is required, go to Step 5; otherwise go to Step 2.

5. Free the policy variables, calculate j_p for all $p=2, \ldots, e$. If $t_{1j_p} < 0$ for some $p=2, \ldots, e$, introduce activity j_p into the basis, fix the level of the levels of the policy variables and return to Step 4. If $t_{1j_p} \geq 0$ for all $p=2, \ldots, e$, stop.

8. *Proof of Convergence*

The algorithm allows for the policy variables to be held at fixed levels, and hence to be reduced in subsequent steps of the algorithm. The same effect could be achieved by imposing upper and lower bounds on the policy variables and "fixing" the policy variables by appropriate changes to these bounds. This latter approach would allow any change of basis to be interpreted as increasing the level of a non-basic activity. It is convenient to discuss convergence on the assumption that all changes in the level of the basic variables result From increasing the level of a non-basic variable (using a positive pivot).

Further, it is convenient to assume that only one activity enters the basis in Steps 2 and 3, and only one activity in Step 4. There is no loss of generality from this assumption since the entry of several activities into the basis can also be represented as the entry of one activity made up of a suitable linear combination of the several activities actually entering the basis.

Step 1 of the algorithm finds a (primal) feasible solution. Primal feasibility depends only on the constraints (3) and (4); hence exactly the same procedure for finding a feasible solution (or proving that one does not exist) can be used for multi-level programming as for linear programming.

Steps 2 and 3 involve bringing in activities which increase the value of the behavioral objective function. In Step 2, the policy objective may increase or decrease, while in Step 3, it will increase monotonically, since in Step 3 the coefficients in the policy and objective functions have the same sign for activities entering the basis.

On completion of Step 3, we have a behavioral optimal solution (all non-basic, non-policy variables have non-negative entries in the behavioral objective row), and in this solution non-basic policy variables with a negative entry in the policy objective have a positive entry in the behavioral objective.

To reiterate, on completion of Step 3 $t_{2j} \geq 0$ $j = 2,\ldots,e$; and for $j = e + 1, \ldots, n$ if $t_{1j} < 0$, then $t_{2j} \geq 0$.

Several non-basic policy variables may be introduced into the basis in Step 4, but for simplicity we assume an appropriate linear combination of these activities is formed, and hence only one iteration is needed using the pivot $t_{rp} > 0$. After this iteration the new behavioral objective function elements can be written

$$t'_{2k} = t_{2k} - \frac{t_{2p}\, t_{rk}}{t_{rp}} \quad ; \quad k = 2, \ldots, e \tag{14}$$

according to the standard rules of Gauss elimination.

Let us then form any linear combination of these non-basic non-policy activities, such that introduction of the linear combination would increase the behaviorial objective function:

$$\sum_{k=2}^{e} \lambda_k\, t'_{2k} < 0 \quad ; \quad \lambda_k \geq 0 \; ; \quad \sum_{k=2}^{e} \lambda_k = 1 \tag{15}$$

Substituting from (14) in (15) we have

$$\sum_{k=2}^{e} \lambda_k\, t'_{2k} = \sum_{k=2}^{e} \lambda_k\, t_{2k} - \sum_{k=2}^{e} \lambda_k \frac{t_{2p}\, t_{rk}}{t_{rp}} < 0$$

where we know

$$\sum_{k=2}^{e} \lambda_k\, t_{2k} \geq 0$$

$$t_{2p} \geq 0$$

$$t_{rp} > 0$$

$$\lambda_k \geq 0 \quad k = 2, \ldots, e$$

Hence

$$-\sum_{k=2}^{e} \lambda_k t_{rk} < 0$$

$$\sum_{k=2}^{e} \lambda_k t_{rk} > 0.$$

Since the t_{rk} refer to tableau elements at the end of Step 3, the corresponding element at the end of Step 4 may be defined as

$$t'_{r*} = \sum_{k=2}^{e} \lambda_k t_{rk} / t_{rp} > 0 .$$

Also, the first action in Step 2 would be to fix the basic policy variable in row r, hence resetting $t_{r1} = 0$. Thus t'_{r*} would be the pivot if this linear combination of activities was introduced into the basis, and it could enter the basis only at a zero level. Thus we can state:

(i) Any linear combination of non-policy non-basic activities introduced into the basis in Steps 2 and 3, following Step 4, which would increase the level of the behavioral objective function, will come in at a zero level.

(ii) And therefore the policy objective increases in Step 4 and (after the first round) does not decrease in Steps 2 and 3.

Step 5 finds a direction (if one exists; in which the basic policy variables can be changed without destroying behavioral optimality. Hence

Step 4 is again entered form a behavioral optimal solution, and the proof of convergence with respect to Steps 2,3 and 4 continues to apply.

Since the number of extreme points is finite, the algorithm must either find a maximum to the policy objective, or show that the problem is unbounded.

In conclusion, it can be seen that relatively minor adjustment of existing linear programming computer codes will permit them also to solve multi-level programming problems.

9. *A Numerical Example*

This section provides a small numerical example of multi-level programming, in which the behavioral subproblem is characterized by maximization of a linear objective function subject to the inequality constraints and non-negativity restrictions of linear programming.8/ The example is based on a well-known problem formulated and solved by Louwes, Boot, and Wage with quadratic programming techniques [1, 11]. By re-casting their problem in multi-level programming, their solution can be assessed in a broader policy context.

In their analysis, the behavioral decision maker is a national cooperative of dairy producers, and the policy decision maker is the Dutch government. Four dairy products are postulated, milk (x_1), butter (x_2), fat cheese (x_3), and 40^+ cheese (x_4), which can be produced subject

8/ Larger, more complex examples are reported and analyzed in a subsequent paper.

to available supplies of fat and dry matter, according to the following technology matrix:

$$
\begin{aligned}
.026x_1 + .8x_2 + .306x_3 + .245x_4 &\leq 121 \text{ (fat)} \\
.086x_1 + .02x_2 + .297x_3 + .371x_4 &\leq 250 \text{ (dry matter)}
\end{aligned}
\tag{16}
$$

Linear demand functions are specified:

$$
\begin{aligned}
x_1 &= 2671 - 1.5413p_1 && \text{(milk)} \\
x_2 &= 135 - .0203p_2 && \text{(butter)} \\
x_3 &= 103 - .0136p_3 + .0015p_4 && \text{(fat cheese)} \\
x_4 &= 19 + .0016p_3 - .0027p_4 && \text{(40}^{+}\text{ cheese)}
\end{aligned}
\tag{17}
$$

Monopolistic pricing is assumed, thus the behavioral objective function is

$$f_1 = \sum_i p_i x_i .$$

Substituting for x_i, we get a problem in the p_i :

$$
\left.\begin{array}{l}
\text{Max } f_1 = 2671p_1 - 1.5413p_1^2 + 135p_2 - .0203p_2^2 + 103p_3 - .0136p_3^2 \\
\qquad\qquad + .0031p_3p_4 + 19p_4 - .0027p_4^2 \\
\text{subject to:} \\
\begin{array}{rrrrcr}
-.0401p_1 & -.0162p_2 & -.0038p_3 & -.0002p_4 & \leq & -92.6 \\
-.1326p_1 & -.0004p_2 & -.0034p_3 & -.0006p_4 & \leq & -20.1 \\
1.5413p_1 & & & & \leq & 2671 \\
 & .0203p_2 & & & \leq & 135 \\
 & & .0136p_3 & -.0015p_4 & \leq & 103 \\
 & & -.0016p_3 & +.0027p_4 & \leq & 19 \\
 & & p_1, p_2, p_3, p_4 & & \geq & 0
\end{array}
\end{array}\right\} \quad (18)
$$

The original authors start by assuming that the price levels themselves arc: policy variables. Simple monopolistic profit maximization leads to unacceptably high price levels, however, and a policy constraint is imposed on the price index for dairy products:

$$.0160p_1 + .0004p_2 + .0005p_3 + .0002p_4 \leq k + 10 \quad (19)$$

Eehaviorally optimal solutions are then obtained for various levels of k, and an optimal pricing strategy is selected by inspection in the light of the resulting product prices, consumption levels, and total revenue. <u>Ceterus paribus</u>, the lower is k, the lower is the "average price level" and the better the solution from the policy viewpoint. A complete policy objective Function is not specified, however, nor are other instruments for influencing the dairy industry admitted.

For the purpose of this exercise, it is convenient to restate problem (13) somewhat. It is now postulated that the government cannot influence the dairy price index directly, but that instead the government may have recourse to a subsidy on liquid milk sales (y_1) and an import duty on butter (y_2). It will be seen that the original Louwes, Boot, and Wage solution is a special case of this slightly more general formulation.

The presence of the subsidy on milk sales means that the revenue to producers from milk sales is now expressed as

$$r_1 = x_1(p_1 + y_1) = x_1p_1 + x_1y_1 \tag{20}$$

instead of $r_1 = x_1p_1$, as before.

Substituting for x_1 in (20) gives the following additional terms in the behavioral objective function:

$$2671y_1 - 1.5413y_1p_1$$

The possibility of importing butter means that total butter consumption is now

$$q_2 = x_2 + x_5 , \tag{21}$$

where x_5 represents the imports of butter.

The price of butter is given by the inverse demand function:

$$q_2 = 135 - .0203p_2 \tag{22}$$

Equations (21) and (22) together yield a new constraint on the behavioral problem, the material balance for butter:

$$x_5 = 135 - .0203p_2 - x_2 ,$$

which replaces the fourth inequality restriction of the original problem (18).

Imported butter may be assumed to be available at an international price of 1500, so that butter is available domestically from import sources at 1500 + y_2, where y_2 is the import duty on butter; and we have the following additional constraint:

$$p_2 \leq 1500 + y_2$$

Taken together, these revisions yield the following restatement of the behavioral optimization problem:

$$\left.\begin{array}{l}
\text{Max } f_1 = 2671p_1 - 1.5413p_1^2 + p_2x_2 + 103p_3 - .0136p_3^2 \\
\qquad + .0031p_3p_4 + 19p_4 - .0027p_4^2 + 2671y_1 - 1.5413y_1p_1 \\
\text{subject to:} \\
\begin{array}{rrrrrrrrrrl}
- .0401p_1 & + & .8x_2 & - & .0038p_3 & - & .0002p_4 & & & \leq & 15.4 \\
- .1326p_1 & + & .02x_2 & - & .0034p_3 & - & .0006p_4 & & & \leq & -17.4 \\
1.5413p_1 & & & & & & & & & \leq & 2671 \\
 & & & & .0136p_3 & - & .0015p_4 & & & \leq & 103 \\
 & & & - & .0016p_3 & + & .0027p_4 & & & \leq & 19 \\
 & & & & p_2 & - & y_2 & & & \leq & 1500 \\
 & & & & .0203p_2 & + & x_2 & + & x_5 & = & 135 \\
 & & & & \multicolumn{5}{r}{p_1, p_2, p_3, p_4, x_2, x_5} & \geq & 0
\end{array}
\end{array}\right\} \quad (23)$$

Problem (23) contains two policy variables, y_1 and y_2, and one new behavioral variable, x_5.

For given values of y_1 and y_2, this is a quadratic programming problem which explicitly includes those policy variables. We must note, however, that this statement of the problem gives no guidance as to desirable levels of the policy variables.

Multiple goals

The essential feature of multi-level programming is that it embeds a behavioral problem, with endogenous behavioral variables (p_1 to p_4 and x_5) within a program aimed at selecting the optimal values for policy variables (y_1 and y_2). To define these optimal values, it is first necessary to explicitly define the goals which policy makers wish to pursue. At least four goals suggest themselves in this case:

i) Since a subsidy on liquid milk production has been assumed, we may equally assume that a low price of milk (p_1) to the consumer is thought desirable.

ii) The original authors indicated a goal of keeping weighted price rises to a minimum.

iii) Other things equal, a high farm income is a sensible goal.[8] And

iv) Since the subsidy entails a cost to the government, while the butter import tax would generate income, policy makers can be expected to be interested in the gross outlays of government revenue implied by policies, even though the butter import tax would generate income.[9/]

Representation of these goals as programming variables (w_1 to w_7) is relatively straightforward:

$$
\begin{array}{lllllll}
w_1 & - p_1 & & & & & = 0 \\
- w_2 & + .0160p_1 & + .0004p_2 & + .0005p_3 & + .0002p_4 & & = 10 \\
+ w_3 & - 2671p_1 & + 1.5413p_1^2 & - 103p_3 & & & \\
 & + .0136p_3^2 & - .0031p_3p_4 & - 19p_4 & + .0027p_4^2 & & \\
 & - 2671y_1 & + 1.5413y_1p_1 & & & & = 0 \\
- w_4 & + 2671y_1 & - 1.5413y_1p_1 & - y_2x_5 & & & = 0
\end{array}
$$

As discussed elsewhere [2], it is extremely unlikely that policy makers' indifference systems are known ex ante, or indeed that they are invariate over time. Thus, in practice, discovery of good policy and definition of the indifference system in the neighbourhood of this "good" policy should proceed simultaneously. For this example, we may also assume that, ceterus paribus, the policy makers prefer:

i) w_1, the price of milk to be as small as possible;

ii) w_2, the change in the price level, to be as small as possible;

9/ In a more sophisticated analysis, the percentage of higher farm incomes returned to the government in taxes could be included in goal (iv); but in the present case, this would unnecessarily complicate an already too complex numerical example.

iii) w_3, farm income, to be as large as possible; and

iv) w_4, net subsidy to the dairy industry, to be as small as pcssible.

where "as small as pcssible" means "as close to minus infinity as possible". It turns out, as shown below, that an interesting set of policy variations is generated by experimenting with weights of ± 1 on the four "impact variables" (goals), w_1 to w_4.

Model Restatement

Gathering the various equations together, the model may be restated now in multi-level programming form:

$$\text{Max } f_2 = d_1w_1 + d_2w_2 + d_3w_3 + d_4w_4$$

subject to:

$$f_1 = 2671p_1 - 1.5413p_1^2 + p_2x_2 + 103p_3 - .0136p_3^2$$

$$+ .0031p_3p_4 + 19p_4^2 - .0027p_4 + 2671y_1 - 1.5413p_1y_1 \quad \text{a max}$$

and

$$\left.\begin{array}{lr}
- .0401p_1 + .8x_2 - .0038p_3 - .0002p_4 & \leq 15.4 \\
1.5413p_1 & \leq 2139 \\
.0136p_3 - .0015p_4 & \leq 103 \\
- .0016p_3 + .0027p_4 & \leq 19 \\
p_2 - y_2 & \leq 1500 \\
.0203p_2 + x_2 + x_5 & = 135 \\
- w_1 + p_1 & = 0 \\
- w_2 + .0160p_1 + .0004p_2 + .0005p_2 + .0002p_4 & = 10 \\
w_3 - 2671p_1 + 1.5413p_1^2 - p_2x_2 - 103p_3 + .0136p_3^2 & \\
.0031p_3p_4 - 19p_4 + .0027p_4^2 - 2671y_1 + 1.5413p_1y_1 & = 0 \\
w_4 + 2671y_1 - 1.5413p_1y_1 - y_2x_5 & = 0
\end{array}\right\} \quad (24)$$

and

$$p_1, p_2, p_3, p_4, x_2, x_5, y_1, y_2 \geq 0$$

where w_1, w_2, w_3 and w_4 are unrestricted impact variables,

and d_1, d_2, d_3 and d_4 are policy makers' relative weights on the values of the impact variables w_1, w_2, w_3 and w_4.

Although problem (24) is quadratic, it was approximated in a linear programming format in order to make it amenable to solution with the algorithm reported in this paper. The approximation involves a grid linearization applied across variables entering into the quadratic terms, and it can be made an arbitrarily close approximation to the original nonlinear problem [7].

The top rows (above row f_2) in Table 4 illustrate the settings of the variables used to construct the convex combination sets of activities for the linearized terms. The row f_2 is the policy objective fucction, while the row f_1 is the behavioral objective function. For given levels of the policy variables, the function f_1 is to be maximized (section 4 above). The control rows ensure that a convex combination of the appropriate activities enters the basis for the linearized terms. The fat and dry matter and import duty restraints are the normal types of linear restraints, the next four rows refer to the four goal functions, and the final row refers to the level of y_2, one of the policy parameters.

The first column in Table 4 gives the level of the restraints (commonly referred to as the "right-hand side"). Columns 2 to 13 refer to convex combination of p_1 and y_1, columns 14 to 26 refer to levels of p_2 and x_5 ; and columns 27 to 38 refer to levels of p_3 and p_4 . Columns 39 to 42 refer to the impact variables, and columns 43 and 44 refer to levels of the policy variables y_1 and y_2 .

Table 4: Matrix of Coefficients for the Sample Multi-level Programming

Column	1 (RHS)	2	3	...	13	14	15	...	26	27	28	...	38	39	40	41	42	43	44
1		350	550	...	950														
1					200														
2						2900	3100	...	2900										
5									.75										
3										2500	2500	...	4500						
.										3000	4000	...	6000						
	a max													-1	-1	1	-1		
1	a max	746041	1002807	...	1387779	220777	223417	...	3277	228450	236300	...	288600						
ontrol $P_1 \cdot Y_1$	1 ≥	1	1	...	1														
$P_2 \cdot x_5$	1 ≥					1	1	...	1										
$P_3 \cdot P_4$	1 ≥									1	1	...	1						
	121 ≥	55.42	47.41	...	31.38	60.90	57.66	...	.9	26.14	25.94	...	18.00						
y Matter	250 ≥	183.3	156.4	...	103.8	1.5	1.9	...	.023	27.4	26.8	...	18.8						
rt Duty	1500 ≥					2900	3100	...	2900										-1
lk Price	0 =	250	550	...	950									-1					
ice Change	10 =	5.6	8.8	...	13.2	1.16	1.24	...	1.16	1.85	2.05	...	3.45		-1				
rm Income	0 =	-746041	-1002807	...	-1387780	-220777	-223417	...	-3277	-228450	-236300	...	-288600			1			
ov't Revenue	0 =				193378				-105000								1		
	0 =				200													-1	

Note: The column variables are described in the text.

For want of better estimates, the policy weights d_1 to d_4 were initially set at the values - 1 -1, 1 and -1, respectively. These sample values are shown as coefficients in row f_2 of Table 4.

Numerical Kesults

In addition to the initial objective of weights of ± 1 on each goal (impact variable), the model also was run eight more times, first maximizing, then minimizing, each impact variable individually. Of the nine solutions, only four were distinct, and these four are summarized in Table 5. Table 6 defines the relationships between the nine policy weight combinations and the four distinct solutions.

The first thing to notice in Tables 5 and 6 is that solution 3, which gives a weight only to the impact variable w_2 (percentage change in the price index for the four products), closely approximates the Louwes-Boot-Wage quadratic solution in which they constrained w_2 to equal 8.35%. The approximation is within about 4% in the price variables (owing to the linearization) and it is exact in terms of the value of the behavioral objective function. While the Louwes-Boot and Wage solutions are certainly technically feasible, and could even be obtained, for example by price control, it is significant to notice that if the government is limited to the twin policy instruments of milk subsidy and butter imports, then the price level change can only take four distinct value?.

This illustrates a point, mentioned elsewhere, but which seems worthy of reemphasis even within the context of this overly long paper, namely that the politically feasible solution set may be much smaller than the technically feasible. If the government was unwilling to introduce

Table 5: Numerical Results for Different Policy Weights

Solution	Policy Variables v_1	v_2	Impact Levels w_1	w_2	w_3	w_4	Behavioral Objective V	Behavioral Variables x_1	x_2	x_3	x_4	x_5	p_1	p_2	p_3	p_4
Multi-level Programming Model																
	670	0	530	2.54	2650	-124.2	2650	1854	91.36	51.21	9.7	0	530	2150	4480	6090
[illegible]	0	0	870	7.98	1583	0	1583	1330	91.36	51.21	9.7	0	870	2150	4480	6090
3	0	.183	870	8.71	1670	0	1670	1330	67.40	51.21	9.7	0	870	3330	4430	6090
4		183	530	3.27	2738	-124.2	2738	1854	67.40	51.21	9.7	0	530	3330	4480	6090
To Wong, Boot and Wage (5, Table 3, p. 317																
1	0	∞		0			1204	2034	54.6	67.7	16.8	0	414	3970	2891	2351
2	0	∞		1			1384	1945	58.4	66.0	15.8	0	471	3781	3064	2630
	3	∞		2			1452	1857	62.3	64.3	14.9	0	529	3591	3237	3273
	0	∞		3			1527	1768	66.1	62.7	13.9	0	586	3402	3410	3727
5	0	∞		4			1578	1680	70.0	61.0	13.0	0	643	3212	3583	4175
6	0	∞		5			1617	1591	73.8	59.3	12.0	0	701	[illegible]	3756	4623
7	0	∞		6			1622	1515	72.0	57.0	11.2	0	750	3112	3973	5028
8	0	∞		7			1662	1439	70.1	54.7	10.5	0	780	3026	4191	5431
9	0	∞		8			1670	1363	68.2	52.0	9.8	0	849	3300	4408	5835
10	0	∞		8.35			1670	1363	67.5	51.5	9.5	0	867	3333	4485	5977
11	0	∞		9			1668	1286	66.3	50.0	9.0	0	899	3394	4626	6238

price controls, then knowing the price levels which would maximize income holding the price rise to 5% is acsdemic in tha bad sense. What is required is a colution which maximises the policy maker's objectives in the light of the policy instruments under his control.

The interaction between the impact variables, the behavioral objective function, and the choice of policy variables is clearly illustrated in the first four rows of Table 5. With neither a milk subsidy nor a tariff (solution 2), farm income would be at a level of 158.3. Use of either policy instrument raises farm incomes. Imposition of a milk subsidy also lowers the milk price and therefore lowers the price index, but incurs treasury costs.

Table 6: Correspondence Between Policy Functions and Solutions

Policy Weights				Optimum Solution*
d_1	d_2	d_3	d_4	
1				1
	1			4
		1		3
			1	1
-1				2
	-1			1
		-1		2
			-1	4
-1	-1	-1	-1	4

* These solution numbers are the ones used in the upper section of Table 5.

These variations with unit policy weights constitute extremely simple policy experiments, and in an actual decision setting many other values would be tried, in consultation with policy makers. Nevertheless, even these simple experiments illustrate the potential of policies to induce varying reactions from the producers. In the case of one product, butter (x_2), the milk subsidy policy induces the butter production and price to lie well outside the range of values explored by Louwes, Boot, and Wage. Thus, even though the "policy objective function" cannot be defined numerically _a priori_, multi-level programming appears to be a powerful tool for systematically exploring the "policy space", in terms of both the impact variables (goals) and the attendant reactions to policies of producers, consumers, and other economic actors.

In more general terms, it always is possible to set up policy problems explicitly so that the variables controlled by policy makers can be distinguished from the variables determined by other (behavioral) decision makers. This, in turn, allows an explicit mapping from the policy variable space, to the space of policy goals, with clear presentation of trade-offs.

In large model systems with many policy variables, the advantages of multi-level programming would appear to be stronger. The traditional procedure is to attempt to enumerate, via successive solutions, a large number of points along each policy variable axis, and also along many combinations of policy axes. With multi-level programming, instead, the policy objective function can be used to define immediately the more interesting policy settings, e.g., those combinations of instrument values which optimize according to at least one definition of the objectives of policy.

in multi-level programming, policies are no longer being constrained only by the technologically feasible frontier, but also by the frontier of behaviorally feasible points, giving due allowance to decentralized decisions in reaction to policies.

10. *Notes on Sufficient Conditions for Convexity*

The algorithm of sections 5 through 8 is designed to be used when the feasible sets for both maximization problems are closed and convex, and when both objective functions are quasi-concave. Multi-level programming is open to the same abuses that mathematical programming is, in the sense of attempted applications when these conditions are not met. The principal danger is that a local optimum may be mistaken for a global optimum. Therefore a few remarks are in order about the convexity conditions.

Normally, it is a fairly straightforward matter to insure convexity of the feasible set for the behavioral problem. It is not always clear, however, that the feasible set for the policy problem also is convex. It should be said first, in the algorithmic spirit of this paper, that there exist unambiguous solutions to the problem of nonconvexities; they involve use of existing features of solution routines for mathematical programming. One way is to declare some of the impact variables integer and to use branch-and-bound methods to combine our algorithm with the mixed-integer procedures. A more elegant method, which does not require certain knowledge of the variables in which the nonconvexities appear, is to utilize features of the new "special ordered sets" commercial algorithms for mathematical programming.

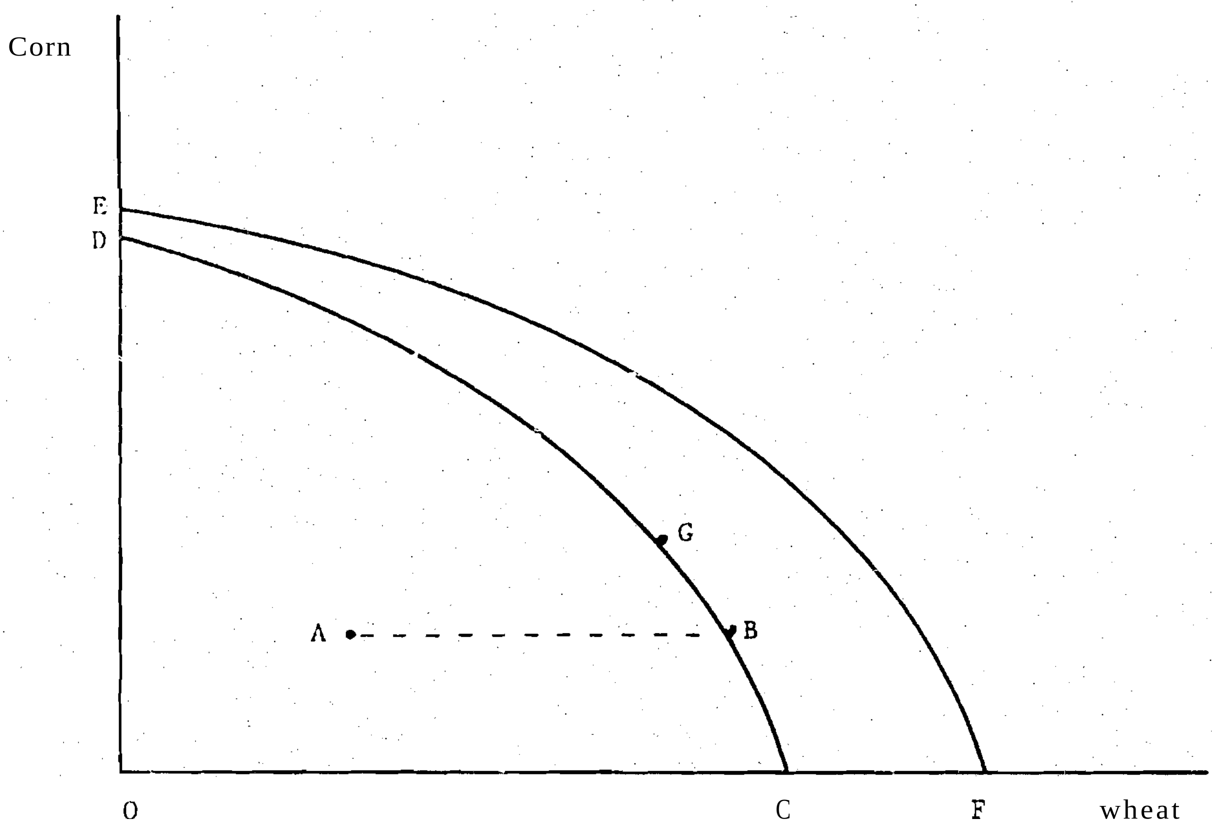

Figure 1: Feasible Sets

Figure 1 illustrates the nature of the feasible sets that we are concerned with. In this example, for the entire agricultural sector the outer frontier EF represents the technological maxima in production: it is the "technology frontier". However, if consumer demands for agricultural products have an elasticity which is less than unity in absolute value, in some range of prices, then producers may not attain maximal. profits by producing at the technological maximum. If their objective function (f_2) is profits, then they nay select a point like A, in the absence of policy changes. It is important to note that, in the absence of market distortions, the point A will lie on the technological frontier (production possibilities frontier) in the full s-dimensional space of behavioral variables $\underline{x}_1^*$. But the space of impact variables $\underline{x}_1$ is of dimensionality $r \leq s$, and in reality it is always the case that $r<s$. In other words, policy makers are interested in only a subset of the totality of economic variables in the world. Hence any projection of the s-dimensional solution onto the r-dimensional hyperplane will almost certainly lie inside the frontier in r-space.

Policy changes can, of course, alter the absolute and relative profitability levels of the two kinds of production shown in Figure 1. With an unlimited ability to subsidize, the government could, very probably, induce producers to move to the technological frontier EF. However, the domain of policy instruments is always restricted, and so the maximum observable levels of production, allowing for varying policy inducements, may be characterized by the frontier CD. The set OCD is the feasible set for the policy problem. Its frontier is defined by (a) the
4 behavioral objective function, (b) the feasible values of polity instruments, and (c) the constraints to the behavioral problem. The frontier may be called the policy-behavioral frontier, for short.

In the example of Figure 1, subsidizing wheat may induce producers to move from interior point A to point B on the frontier of the policy problem's feasible set. A subsequent redeploymect of policy instruments in the direction of corn incent'ves would then move producers along the policy-behavioral frontier toward point G. The convexity requirement is that the marginal rate of substitution of corn for wheat along that frontier be negative and decreasing.

In other words, once at point B, then with the most efficient instrument for inducing more corn production,[10/] there must be decreasing returns in corn per unit of wheat given up. This is a reasonable requirement.

If we define a set of reaction functions:

$$y_j = h_j(\underline{u}) \quad \text{all } j \tag{25}$$

where y_j is the level of the jth impact variable, and

$\underline{u}$ is the vector of policy variables,

then a very strong sufficiency condition would be that the Hessian of partial derivatives

$$H_j = \left[\frac{\partial^2 y_j}{\partial u_i \, \partial u_k} \right] \quad \text{all } j \tag{26}$$

be negative semi-definite. Unfortunately, in many cases condition (26) cannot easily be ascertained by ex ante examination of problem structure. Failures to meet (26) may be readily evident; but absence of failure to meet the sufficiency condition is difficult to establish. (Two examples where (26) does not hold are given below.)

Using the notation of section 4, the Kuhn-Tucker necessary conditions for a maximum to the one-level problem can be written:

10/ With instruments other than the most efficient one, the new solutions would Lie at interior points in the set OCD.

That

$$L_1 (\underline{x}_1, \underline{\lambda}_1 \mid \underline{x}_2) = f_1 (\underline{x}_1 \mid \underline{x}_2) + \sum_i \lambda_i g_i (\underline{x}_1 \mid \underline{x}_2) \tag{27}$$

be at a saddle point. That is, $\underline{x}_1^o$ is an behavioral optimum solution (one-level optimal) to the one-level problem if, and only if, there exist non-negative multipliers $\underline{\lambda}_1^o$, such that:

$$L_1 (\underline{x}_1, \underline{\lambda}_1^o \mid \underline{x}_2) \leq L_1 (\underline{x}_1^o, \underline{\lambda}_1^o \mid \underline{x}_2) \leq L_1 (\underline{x}_1^o, \underline{\lambda}_1 \mid \underline{x}_2) \tag{28}$$

The Kuhn-Tucker necessary conditions for a saddle-point are:

$$\left.\begin{array}{ll} \underline{x}_1^o \geq \underline{0} & \underline{\lambda}_i^o \geq \underline{0} \\ \left[\dfrac{\partial L_1}{\partial x_{1,j}} \right]^o \leq \underline{0} & \left[\dfrac{\partial L_1}{\partial \lambda_{1,i}} \right]^o \geq \underline{0} \\ \underline{x}_1^o \left[\dfrac{\partial L_1}{\partial x_{1,j}} \right]^o = 0 & \underline{\lambda}_1^o \left[\dfrac{\partial L_1}{\partial \lambda_{1,i}} \right]^o = 0 \end{array}\right\} \tag{29}$$

Given this same notation, we can write the two-level problem as

$$f_2 (\underline{x}_2) \quad \text{a max} \tag{30}$$

$$\text{s.t.} \quad f_1(\underline{x} \mid \underline{x}_2) - f_1(\underline{x}_1^o \mid \underline{x}_2) \geq 0 \tag{31}$$

$$g_i(\underline{x}_1 \mid \underline{x}_2) = 0 \tag{32}$$

where $\underline{x}_1^o$ satisfied the Kuhn-Tucker necessary conditions for the one-level problem. Following Kuhn-Tucker, we can say that a necessary condition for the two-level problem to be a maximum is that:

$$L_2(\underline{x}_1, \underline{x}_2, \underline{\lambda}_2) = f_2(\underline{x}_2) + \lambda_o\Big[f_1(\underline{x}_1 \mid \underline{x}_2) - f_1(\underline{x}_1^o \mid \underline{x}_2)\Big] + \sum_i \lambda_i \, g_i(\underline{x}_1 \mid \underline{x}_2) \tag{33}$$

be at a saddle-point. That is $\underline{x}_1^o, \underline{x}_2^o$ is a policy optimal (two-level optimal) solution if, an2 only if, there exist non-negative multipliers $\underline{\lambda}_2^o = \{\lambda_{2,i}^o; \ i = 0, I, \ldots, m\}$, scch that

$$L_2(\underline{x}_1, \underline{x}_2, \underline{\lambda}_2^o) \leq L_2(\underline{x}_1^o, \underline{x}_2^o, \underline{\lambda}_2^o) \leq L_2(\underline{x}_1^o, \underline{x}_2^o, \underline{\lambda}_2) \tag{34}$$

The necessary conditions for this saddle-point can now be written.

$$\left.\begin{array}{ll} \underline{x}_1^o, \underline{x}_2^o \geq \underline{0} & \underline{\lambda}_2^o \geq \underline{0} \\[2ex] \left[\dfrac{\partial L_2}{\partial x_{k,j}}\right]^o \leq \underline{0}, \quad k = 1,2; & \left[\dfrac{\partial L_2}{\partial \lambda_{2,i}}\right]^o \geq 0 \\[4ex] (\underline{x}_1, \underline{x}_2)\left[\dfrac{\partial L_2}{\partial x_{k,j}}\right]^o = 0 & \underline{\lambda}_2 \left[\dfrac{\partial L_2}{\partial \lambda_{2,i}}\right]^o = 0 \end{array}\right\} \tag{35}$$

The proof follows the same form as the Kuhn-Tucker one-level proof.

Two examples of non-convex policy problems, due to early critics of this paper,[11/] will serve to emphasize that the convexity requirements of the algorithm are non-trivial.

A Non-Linear Non-Convex Problem

Suppose the policy makers' objective function is

$$\max_{W} U = f_2(V, W) \tag{36}$$

and the behavioral objective is

$$\max_{L} Z = X(a - \tfrac{1}{2} bX) - WL \tag{37}$$

with

$$X = L^{\alpha} \tag{38}$$

$$P = a - bX \tag{39}$$

$$V = PX - WL \tag{40}$$

The behavioral objective may be thought of as the competitive market maximand [7], with

X = output

L = labor input

P = price of the output

V = profits in production

W = wage rate

11/ Clive Bell, Shantayanan Devarajan, and Pasquale Scandizzo.

we have

$$\frac{\partial Z}{\partial L} = \alpha a L^{\alpha-1} - \alpha b L^{2\alpha-1} - W = 0 \qquad (41)$$

Now if $\alpha = \frac{1}{2}$, then by (41),

$$L = \left(\frac{a}{2W + b}\right)^2$$

and
$$X = L^{\alpha} = \left(\frac{a}{2W + b}\right)$$

Therefore
$$V = 2W(1 - \alpha)\, a^2 / (2W + b)^2 = kW/ (2W + b)^2$$

Thus the policy-behavioral frontier in (V,W) is non-convex. Some portion of this non-convexity may be attributable to a partial-equilibrium model specification, where demand is assumed independent of income. Nevertheless (37) to (40) is the sort of model sometimes constructed in partial analyses.

If the impact variables were (V,X) or (V,L), the sufficient conditions would be satisfied in the relevant ranges. Figure 2 shows the approximate shape of these functions, and owing to (weak) concavity of the policy objective function, the areas to the left of and below the dotted lines are irrelevant.

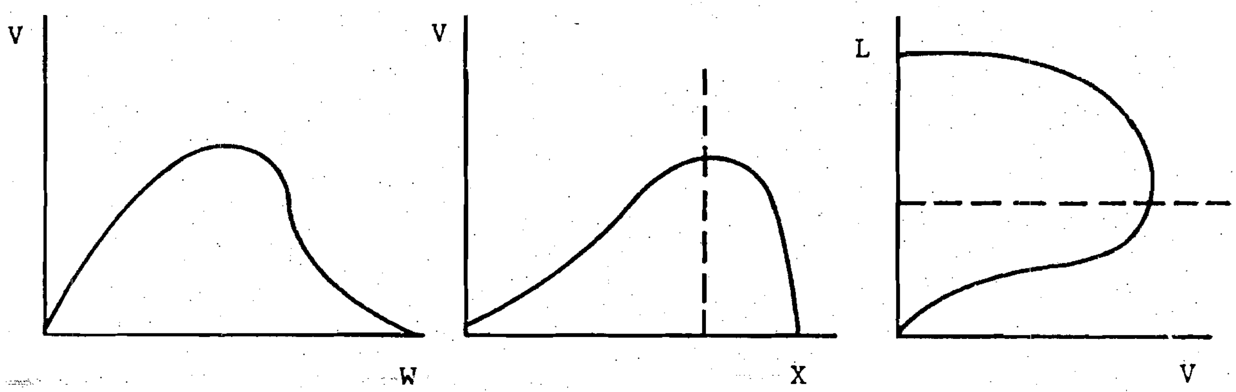

Figure 2: Policy-Behavioral Frontiers

Numerous comments can be made about the realism of this example, alternative solution procedures, etc. The point is that, as stated in (36) - (40), with $a = \frac{1}{2}$, it does not fit into the algorithm of sections 5-7. The same is true of the other illustration below.

A Discontinuous Problem

Thi:: counter-example concerns an agricultural landlord and his tenant farmer. The tenant can allocate his five acres of land between cotton and beans. Yields are constant and revenue for each crop is proportional to the acreage in that crop. An acre of cotton yields \$100 while an acre of beans yields \$60. The landlord can set the share a of cotton revenue to be paid to him, but he receives no income from beans. Assuming both landlord and tenant attempt to maximize revenue, what should they do? If the landlord sets $a > 0.4$, the tenant grows only beans.

Hence the landlord would like to set a exactly at 0.4. But unfortunately at that point the tenant is indifferent between growing cotton and beans, and so the solution is indeterminate. Again, the algorithm does not apply.

11. Conclusions

This paper has defined a new class of "multi-level programming" problems, of which mathematical programming is a subset (i.e. it is one-level programming). One concrete and relevant field of empirical application relates to economic policy problems, where decision makers can bc arranged in some sort of heirachical order and, for lower-level decision makers, the variables under the control of higher-level decision makers can be taken as given.

An algorithm and numerical example have been given for a very simple case, namely where all functions are linear, and both the one- and two-level problems are convex.

Perhaps the most important contribution of this paper, however, is to provide a formal, potentially quantitative, framework for the analysis of policy problems. It places in high relief the questions: who controls which variables? And, what motivates them to prefer one set of values to another? This approach gives precise content to the frequent observation that just because it is technically possible does not mean it is politically possible.

This observation also relates to the use of shadow prices in project evaluation, e.g., the case where there is severe unemployment and a wage rate less than the market wage is used to evaluate project costs. The logic of this approach is impecable, where the government does indeed control the investment decision; but if private investors are involved they may not proceed unless the project is viable at the narket wage. Multi-level programming allows us to address the question: what level of our policy instruments will indeed encourage private investors to proceed in the way desired by government policy makers, who may place a high subjective weight on employment generation.

We have also emphasized that while the distinction between policy and behavioral (two- and one-level) variables has been known for many years, quantitative models explicitly reflecting this knowledge have not been built. This means that we are into a new field, so that the solution characteristics of even two-level problems are not well known. We have, however, shown that even where the one-level problem is linear and convex (a linear programming problem, in fact), and the two-level problem involves continuous variables, yet the solution space of the two-level problem may not be convex, and it may not be continuous either.

It is the authors' belief that the more complex the problem, the more important it is to have a formal analytical framework within which the problem can be explicitly defined. This paper is intended to be a modest step toward expanding the range of problems which can be expressed in formal frameworks.

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