GLOBAL PARTNERSHIP for EDUCATION

Teaching mathematics effectively to primary students in developing countries: Insights from neuroscience and psychology of mathematics

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GPE Working Paper Series on Learning, No. 5

December 30, 2008

Executive Summary

This paper uses research from neuroscience and the psychology of mathematics to arrive at useful recommendations for teaching mathematics at primary level to poor students in developing countries. In general, the recommendations are as follows:

Implications of Inherent Number and Geometrical Sense

At all levels build students' number line and focus on increasingly sophisticated training to assess amount, order, space, time and value (or money). Build the capacity to estimate approximately. Focus on both non verbal spatial understanding and language and symbolic reasoning. Build on the inherent geometrical sense by telling shapes apart, combining manipulatives to create new shapes. Use board and computer games.

Early Formal Schooling

Pre-primary education to include nutrition for brain development: Instruction in building a number line: counting, learning correspondence of names and symbols, measuring with tape, estimating approximate quantities, playing board games like snakes and ladders. We recommend pre-primary education including: Stimulating environment, training parents to help students, providing good nutrition

Working Memory and automaticity

We recommend taking into account the limits of working memory to facilitate problem solving. Students should memorize multiplication tables and frequent additions to the point of fluency. Teach the students various mnemonic strategies and also use visual aids. For problem solving, reading fluency is essential. Routines and automaticity are of vital importance in the learning process

Metacognition

We recommend developing students' ability to monitor their own learning processes (metacognition and metamemory) in order to enhance the learning and reasoning skills. For example, students should be asked how they followed a line of reasoning.

Recommended lesson structure

Begin lessons with a review of previous material, introduce new concepts, and leave enough time for students to have individual or group practice.

Teach students by using imagery and gestures to delineate space and sequence; analogies, solved examples, show and tell at the same time. Develop conceptual understanding by using all parts of the triple code (through explanations, gestures, and imagery) rather than focus only on numbers and going through procedures. Distribute practice of exercises through weeks and months, even after topics have changed.

Recommended instructional hours per school year

1,200 hours in upper middle-income countries to a minimum of 1,140 hours in high-income non-OECD countries; is the norm. Because of challenges in teaching math in developing countries, we recommend allotting more hours to provide time for review and practice; Indonesia reports 1,755 hours per school year.

Homework

Give at least 15 minutes of homework per night; homework that is frequently assigned but not lengthy has been shown to be positively correlated with

mathematics achievement. Particularly pay attention to homework assignments by the weakest and strongest students.

Workbooks

If possible, students should have their own one-use workbook as this has been shown to increase achievement.

Manipulatives

We recommend minimizing the use of manipulatives that are highly concrete and rich in perceptual detail, such as toys or other manipulatives that are familiar and can distract from the qualities needed for the lesson

Use manipulables that demonstrate geometric shapes and can be combined for new groups or shapes

Group work

Group work is recommended when students are given challenging problems to solve; routine problems are best solved individually.

We recommend a heterogeneous group if the tasks need input from different perspectives, and the subject is new to all. Students who are gifted should be engaged in learning at their level (sometimes with other advanced students) in class rather than be used as classroom tutors.

Group work is most successful when teachers provide clear rules and expectations of students and how they are to interact in groups.

Computers and calculators when students have mastered basic skills

Used correctly, they aid in the learning of mathematics

The students become more competent users of general computer technologies.

Do not use calculators before the students have some basic skills

Learning in another language

We state that learning-wise, it is beneficially to learn in one's own language

However, being fluent in English helps prepare students for the global marketplace – hence encourage bilingualism

Benchmarks for K-6

At the end of primary schools, students should be proficient in the basic 4 numerical operations, in decimals and fractions and be fluent in these calculations.

Different benchmarks can be established for various grades: e.g. by the end of grade 2, children should display quick recall of addition facts and related subtraction facts and fluency with multidigit addition and subtraction. (US National Council of Teachers of Mathematics)

We recommend the use of curricula similar to what is used in Singapore, or countries using the "A+" curricula (Annex B).

We cannot encourage very reformed-based/progressive curricula as there is no compelling evidence showing positive student outcomes.

Curricula for K-6 in numbers and geometry

We recommend the use of curricula similar to what is used in Singapore, or countries using the "A+" curricula (Annex B).

We cannot encourage very reformed-based/progressive curricula as there is no compelling evidence showing positive student outcomes.

Testing

We recommend the use of formative assessments as a way of monitoring student progress, refining lesson planning, and giving students more opportunities to learn.

We provide three general principles for evaluating mathematics assessments.

Dyscalculia

We recommend a relaxed, welcoming, empathetic, and low-stress atmosphere

Teach repetitive additions

Use instructional technology remediation programs and aids

Encourage practice

Use representational systems to facilitate learning

Encourage students to verbalize their perception of the arithmetic procedure while the teacher provides feedback

Comments on teacher training

We recommend a strong focus on content for primary teachers and automaticity acquisition for fluent and correct calculations in order to monitor students' performance.

Focus on the Pedagogical Concept Knowledge (PCK) which integrates content and pedagogy

In situations where teaching shortages limit the number of highly qualified primary teachers who are proficient in mathematics, we recommend adopting a mathematics teaching specialist program. Mathematics teaching specialists are professionals who would provide on-site professional development and lesson planning for primary grades teachers. These programs have been shown to increase student achievement scores in mathematics

1. Introduction: The Challenges of Teaching Primary School Mathematics in Developing Countries[1]

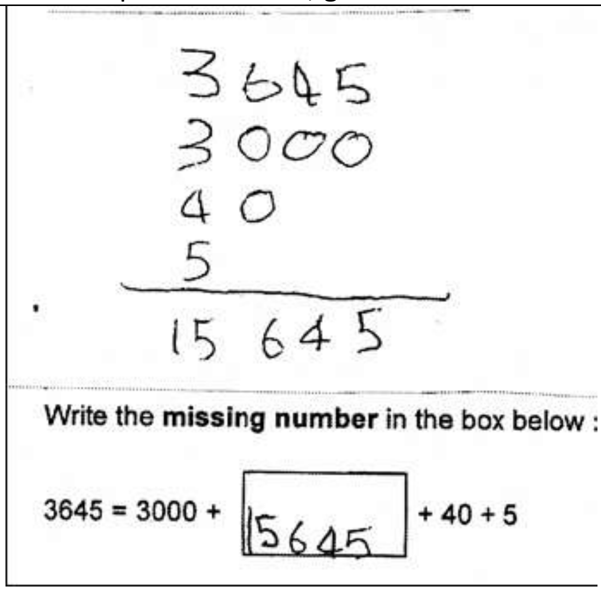
The enrollment rates of the poorer students have improved tremendously in the last decade. And the the global NER (net enrollment ratio) has improved since 2001 from 83.2% to 90-95% except in Sub-Saharan Africa and South Asia. However, this enrollment explosion has had quality consequences. In developing countries, the teaching situation is dominated by “large classes, lack of quality materials, poor physical conditions, and insufficient teacher qualifications” (Skott, 2005, p. 1). Making teaching of math and other subjects efficient for the poor in developing countries is a great challenge, particularly in south Asia and sub-Saharan Africa.

Many developing countries have explored new means of teaching math and other subjects. For instance Eritrea introduced in 2001 ‘a student centered approach’ to replace ‘passive listening’ and ‘didactic and traditional pedagogy’ with more interactive and participatory teaching-learning styles, but contextual factors limit the opportunities for the rhetoric to play prominently in practice” (Skott, 2005, p. 1). Mongolia changed its mathematics education, aiming to build a new set of priorities and practices, given the abandonment of earlier traditions” (Skott, 2002, p. 105).

Similar to international trends of the time, South Africa in the 1990s extensively applied the constructivist learning philosophy which relied on exploration and discovery, with little emphasis on memorization, drill, In conformity with a belief that teachers could develop their own learning programs, there was virtual absence of a national or provincial syllabus or textbooks. Students were expected to develop their own methods for arithmetic operations, but most found it impossible to progress on their own from counting to actual calculating (see examples in Figure 1). According to Schollar (2008), 79.5% of Grade 5 and 60.3% of Grade 7 children still rely on simple unit counting to solve problems to one degree or another, while 38.1% and 11.5%, respectively, of them rely exclusively upon this method.

Figure 1: Outcomes of discovery-based curricula for math instruction in South Africa

[1] We would like to acknowledge the following people: First and foremost we would like to thank Helen Abadzi (World Bank). We would also like to give thanks to Lisser Rye Ejersbo (University of Aarhus, Denmark), Jeppe Skott (Växjö University, Sweden), and Jimmie Fortune (Virginia Tech, USA) for their assistance. The faults remain our own.

An example of multiplication, grade 5	An example of addition, grade 6
	
Source: Schollar 2008, p. iii	

Fortunately, research has shown that it is possible to teach students explicitly and quickly remedy the situation. The Primary Mathematics Research Project (Scholar, 2008) of with 7,028 and 40 schools focused on numbers, operations, and relationships, and assessment standards for grades 3-6, provided a complete syllabus, direct instructions in combination with regular daily exercises and memorization as well as regularly formative and summative assessments over a period of 14 weeks. With 80% curricular coverage of at least 11 weeks, student scores nearly doubled over baseline compared to control groups. Grade 4 and 6 had an increase of 50 and 64%, respectively.

These examples illustrate the importance of informing classrooms from the findings of learning research and also empirical research. The paper is a first effort dedicated to this goal.

2. Methodology

We did a subject specific search through search engines such as ERIC in order to find recent and relevant publications. We have also asked colleagues in the field for advice on authors and literature as well as “snowballing” – i.e. looking in the list of references of already found literature. We do not presume that we have read and found *everything* but we do believe that we have found a range of relevant literature that makes us able to give a covering picture of research results as we as give useful recommendations. We believe that hhaving a basic understanding of the neurological underpinnings of cognitive development as it relates to mathematical thinking can foster instructional efficiency. Specifically, an understanding of neurological development helps educators to better plan developmentally appropriate mathematics instruction. Such instruction would, in turn, reduce instructional wastage resulting from attempting to introduce concepts to a child before a child is ready to learn them.

This study integrates pertinent research from neuroscience and the psychology of mathematics to arrive at recommendations for curricular and efficient means of mathematics instruction particularly for developing countries and poor students at primary level. Specifically, the latest research in neuroscience, cognitive science, and discussions of national benchmarks for primary school mathematics learning, form the basis of our recommendations. These recommendations have a reasonable chance of working in the situational contexts of developing countries, with their traditions and resources. The suggestions are intended to be precise and readily adopted by teachers. We have aimed at some level of detail in order to explain the research results without assuming that the reader has a thorough understanding of neuroscience or the psychology of mathematics. We encourage the reader to use the list of references in order to have more details.

As per the terms of reference (see Annex A), the following issues are covered:

- mathematics learning and brain research, in particular inherent number sense, and inherent geometry sense.
- key cognitive skills used in mathematics learning such as working memory, automaticity and metacognition.
- specific mathematics teaching practices: the role of pre-primary education, lesson structures, recommended hours of mathematics instruction, homework guidelines, use of workbooks, concrete materials such as manipulatives and board games, recent research on individual and group learning, computer and calculator use, and learning mathematics in a language other than one's mother tongue.
- national curricular benchmarks for primary mathematics learning reviewed in an attempt to distill 'best curricular practices' (Annex B).
- This is followed by a short comment on formative diagnostic tests that can be used to assess various mathematics learning benchmarks.
- Some aspects of teacher education
- dyscalculia[2].

3. Neurocognitive Perspectives on Mathematics Education

This section focuses on recent neuroscience and mathematics psychology research in order to gain some insights into how to teach in a way that is likely to result in successful learning for primary students – in particular those in developing and poor countries.

There is a popular idea that all students are special and different. However, evidence does not support this belief. (We are not talking about differences caused by brain damage, but how the "usual" way of learning is for "most" young people.) Mathematics does not differ fundamentally same across cultures, countries, or gender. Dehaene, the French neuroscientist who has research math most extensively, disputes "the idea that all children are different, and that they need to discover things their own way" "I don't buy it at all. … I believe there is one brain organization" (cited in Holt, 2008). Ridley (2004) also states that "you can invent any and every culture with the same brain. The difference between me and one of my African ancestors of 100,000 years ago is not in our brain or genes, which are basically the same, but in the accumulated knowledge made possible by art, literature and technology (p. 228).

On the other side, many studies have documented various "learning styles" such as 'inchworm'/'part-to-whole' versus 'grasshopper'/'whole-to-part' (Chinn & Ashcroft, 2007) and

[2] The authors would like to note that they contributed equally to this paper.

auditory, visual, kinesthetic, tactile (Winebrenner, 2001). We believe that although our brains are very much the same, every single person is neither the same as everyone else, nor are they disjunct. For instance, as this paper will show, children with dyscalculia need another type of teaching than other students *inter alia* since their brain is wired differently. Hence, there are a lot of common traits in how we learn mathematics, which will be the things we focus on in this paper.

Neuroscience might shed counterintuitive light on learning and that "psychology is an important mediator of brain science, and has its own implications for education" (Blakemore & Frith, 2005, p. 9). These authors go on to state that it is time to explore the lessons for education we can learn from brain science, but also that: "Many neuroscientists question whether we know enough about the developing brain to link that understanding directly to instruction and educational practice" (Blakemore & Frith, 2005, p. 22). Holt (2008, p. 3) refers to Dehaene for saying that "We need psychology to refine our idea of what the imagery is going to show us. That's why we do behavioral experiments, see patients. It's the confrontation of all these different methods that creates knowledge". Also Burgess and O'Keefe (2003) argue that cognitive psychology and systems neuroscience in combination have the potential to provide a neuronal-level understanding of human behaviour.

The neuroscience of mathematics learning and the psychology of mathematics learning have produced soundly conducted research, giving empirical evidence for learning theories and even common-sense ideas about what it means to learn mathematics.

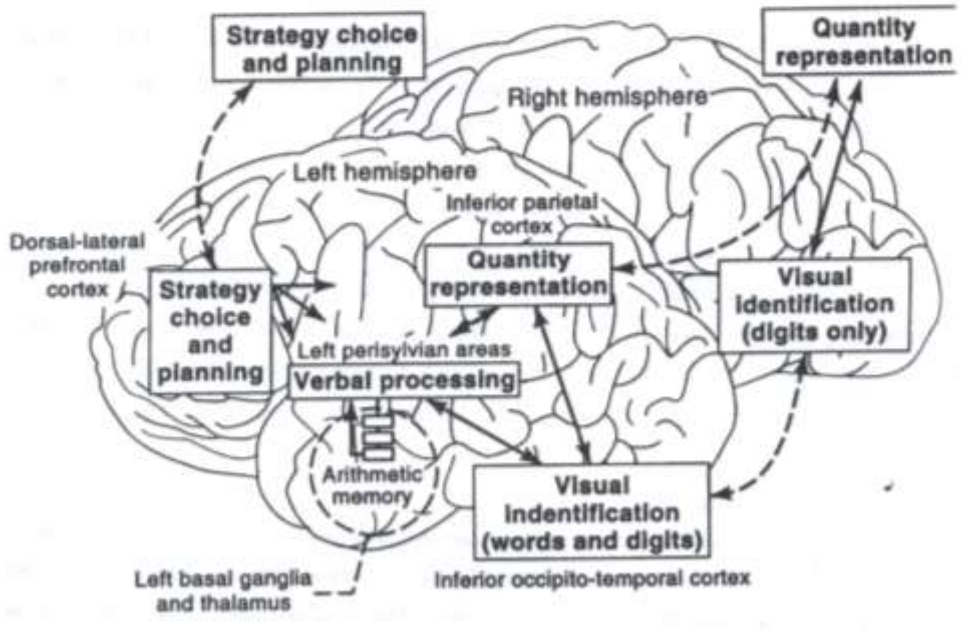
4. Implications of neuroscience and psychology of mathematics for teaching

In this section we will investigate literature on current understandings about learning mathematics as well as the inherent number sense.

Areas in the brain that are related to mathematics

The areas in the brain that "do mathematics" are fragmented into specialized systems (Dehaene 1995). Both sides manipulate Arabic numbers and numerical quantities, however only the left side has access to linguistic connections and a verbal memory of arithmetic tables (Dehaene, 1995). Furthermore it seems that the right hemisphere "approximates" while the left hemisphere calculates precisely (Blakemore & Firth, 2005).

The inputs and outputs of these areas must be integrated in order for people to estimate and calculate correctly. Lack of integration is one potential explanation for dyscalculia, the difficulty that some children have in math performance.



Recent literature in the neurosciences has pointed to structures in the brain that are dedicated to mathematical thinking and reasoning (Deheane, 2001; Micheloyannis et al., 2005; Souza, 2008). For example, Dehaene et al. (2004) found that doing basic arithmetic is associated with enhanced neurological activity in the left parietal lobe of the brain. On that basis he argued (1995) that people manipulated numbers through three channels:

- The see a number as a visual digit (for example, "3");
- They hear or read the number as a word ("three")
- They represent it as a quantity (e.g. "3 is bigger than 1")

Also, brain imaging studies have shown that mathematical thinking can be described in terms of two distinct but interrelated components: a non-verbal spatial understanding of quantity and a 'verbal' understanding that is related to language and symbolic reasoning (Dehaene et al., 1999).

Inherent Number Sense

It seems that some mathematics knowledge is inborn (Blakemore & Frith, 2005, pp. 51-52). Babies are able to add and subtract small sets and understand the mathematical concepts of "more" and "less" that something else. Dehaene (1997) explained that young babies can 'count' in the sense that they are able to recognize when a single object is replaced by several similar objects. These behavioral findings have been substantiated by brain imaging research conducted by Izard et al. (2008). People have a native sense of math, as demonstrated in studies with various indigenous peoples as well as Brazilian street vendors. Though illiterate vendors are often able to make accurate accounts based on their number sense, the ability is rather limited. People are not biologically designed to command large numbers, after the first few, quantities are approximate. More advanced mathematics such as carrying, borrowing, multiplication, division etc. are "unnatural" and must be learned.

Researchers have described an inherent "number sense", which refers to the ability to recognize change in amount from a collection of objects when objects have been added or subtracted without the participant’s knowledge (Sousa, 2008). A recent study demonstrated that this non-verbal, intuitive number sense—or “non verbal math acuity” is correlated to achievement in school mathematics (Halberda et al., in press).

Piaget believed that children do not develop any kind of number sense before they are around 4-5 years old and that learning of mathematics or arithmetic before the age of 6 is rote and without a deep understanding. However, recent research has also shown that children down to the age of 2 can conserve numbers is they are asked to choose between rows containing M&M’s chocolate (Blakemore & Frith, 2005, pp. 49-50). Also Wynn (1992) showed that human babies down to 5 months old have some understanding of basic arithmetic such as 1 + 1 = 2. She investigated this using a Mickey Mouse-doll hidden behind a screen. When she added a second doll behind the screen and then subsequently removed the screen, sometimes two dolls were revealed, sometimes only one. The infants looked systematically longer at the wrong result ‘1+1=1’ that the correct one ‘1+1=2’, which suggested that they had expected two dolls.

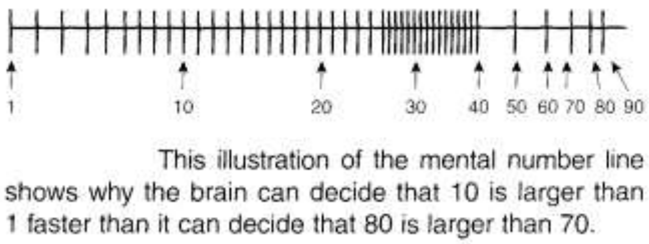
The mental number line

Part of having number sense is the ability to compare numbers. This ability is rooted in our mental construction of number, also known as the mental number line. Sousa (2008) gave the following conclusions based on research on the mental number line conducted by Dehaene et al. (1990), Temple and Posner (1998), Nuerk et al. (2004), and Brannon (2003):

- The amount of time it takes to compare two numbers depends on the distance between them and their size. It takes longer to decide that 12 is larger than 11 than to decide that 3 is larger than 2.
- It takes a longer amount of time to decide that a number is larger than another number for numbers that are close together than it is to decide on the larger of tow numbers that are far apart. For example, it takes a shorter amount of time to recognize that 99 is larger than 36 than it is to decide that 99 is larger than 97.

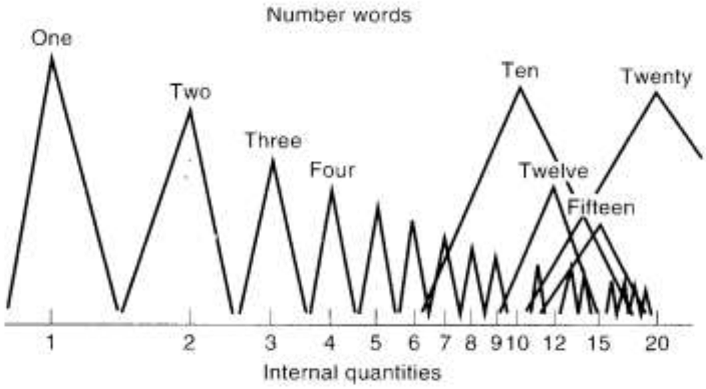
Sousa (2008) explained that these findings hold true because numbers in our mental number line are not evenly spaced. “Instead, the farther we go along the mental number line, the closer the numbers appear to be … as a result, the speed and accuracy with which we carry out calculations decrease as the numbers get larger” (pp. 22-23).

Figure 3: Illustration of a mental number line



Source: Sousa, 2008

Figure 4: Illustration of a mental number line with number words



The decreasing frequency of numerals is due to the organization of our mental representation of quantities. The larger the number, the less accurate our mental representation of it; hence, the less often we need to use the corresponding word. As for round numbers like 10, 12, 15, or 20, they are uttered more frequently than others because they can refer to a greater range of quantities.
Source: Dehaene, 1997, p. 114

The relationship between the number sense and math achievement is very strong. Ability to quickly estimate how many things are in a group significantly predicts school math performance all the way back to kindergarten. Teenagers who did well on a test that measured their "number sense" were much more likely to have gotten good grades in math (Halberda et al. in press).

An innate geometry sense

It seems that people not only have an inherent number sense, they also have an inherent geometrical sense. Dehaene et al. (2006) investigated the knowledge of geometry in an Amazonian Indigene Group. Their results showed the existence of geometrical intuition even in the absence of schooling, experience with graphic symbols or maps or a rich language for geometrical concepts. Adults and children from the American and the indigenous groups were compared. The indigenous children and adults performed at a similar level as the American children while the American adults performed significantly better. However, the American adults made mistakes and the study showed high correlations between the performances across test items between the American and indigenous adults and also between the two groups of children. “Those results again point to a shared pattern of core geometrical knowledge despite increases in absolute performance levels in the educated American adults” (2006, p. 384). In fact Keller, (2004) showed that pre-historic man 40,000 years ago already created geometrical forms.

The studies also show that schooling in geometry matters, i.e. the schooled adults performed better than the unschooled and also an effective teaching program can make young students perform better than older students. The U.S. National Research Council (NRC) (2000, p. 12) report of a study that showed that 2nd grade students having been taught using ‘cognitive guided instruction’ (CGI) in geometry outperformed a control group of undergraduate university students in terms of skills in representing and visualizing three-dimensional forms. CGI is an integral program focusing *inter alia* on the development of children’s mathematical thinking, instructions that positively influence such development and teachers’ knowledge and beliefs that influence such practice (Carpenter et al., 1999).

Implications of inherent number sense research for teaching

1) *Introduce mathematics at the preschool level:* Using findings from brain research, Clements (2001) reasoned that mathematics should be taught at the preschool level as children have the potential to grasp mathematical skills beyond tasks such as practice with addition and subtraction. Preschool should teach not only counting but measurement and approximate estimation.

2) *Engage both components of mathematical thinking:* Activities should focus on both non verbal spatial understanding and language and symbolic reasoning. With regard to non verbal, spatial understanding, correct use of concrete or virtual manipulatives is recommended (see section 5.1 on concrete materials). Also, symbolic reasoning using mathematical symbols rather than words is an important pedagogical practice. Schwartz and Varma (in press) have shown that when it comes to transferring mathematical knowledge from one situation to another, children who learned to think about a mathematical problem using mathematical symbols could more readily transfer this learning to other situations than children who reasoned through mathematical problems using words.
3) *Importance of a stimulating mathematics curriculum:* Teachers can take advantage of the innate number sense by creating activities that are mathematically challenging. This is especially critical among poor children who do not have access to activities at home that help to "formalize" number sense. Analogies are particularly important in providing students with a scaffold to step onto similar concepts and transfer learning to them (Richland et al. 2007).

4) *Structure primary mathematics lessons by numerical concepts, specifically amounts, order, space, time and value.* Rocha (XXXX)[3] reasons that we have inherent number sense with regard to amounts, order, space, time and value and informal number sense, based on experiences before the child enters school. As such, lessons should be structured to illustrate each of these distinct concepts, even though the same numbers are used to represent these different ideas. In doing so, connections could be made between inherent and informal mathematical knowledge and formal mathematical knowledge. Below is an illustration of standards based on the concepts of amount, order, space, time, and value for the 5th level of primary school mathematics:

Example of Grade-Level Development of Basic Math Features: Level V
Amount Dividing an object in parts, the child must be capable to identify the fractions that represent each subgroup of the whole. In level V, the child must be capable of deciding situations that involve the 4 mathematical operations using numbers in the thousands. **Order** Now the child must be capable of using, at minimum, 4 operations involving amounts, space, time and money. **Space** In relation to space, the child must work the 4 operations involving thousands to measure and to calculate

[3] Translated from Portuguese by the author Cachaper.

lengths. Children must be capable of solving situations with 2 operations integrating space with time and amount, involving the 4 mathematical operations. The child must also be able to initiate the conversions of measures between centimeter, meter and kilometer.
Time
In relation to time, the child must be able to work through the 4 operations involving sets of ten to measure and to calculate the times of the day, the week and the month. Children must be capable of solving situations with 2 operations integrating time with space, involving the 4 mathematical operations. The child must also initiate the conversions of measures between seconds, minutes and hours.
Value/Money
In relation to money, the child must work the 4 operations involving thousands to measure and to calculate the values. Children must be capable of solving problems with 2 operations integrating money with amount, involving the 4 mathematical operations. The child must also initiate the conversions of measures between cents—or fractions of the whole currency—and the whole currency.

5. Pre-primary teaching and learning

In human babies at the age of 2-3 months there is a rapid increase in the number of synaptic connections created (*synaptogenesis*) in the *visual cortex*, which is the area that makes sense of visual stimuli. It peaks at the age of 10 months from which a steady decline begins that stabilizes around the age of 10 years and then remain the same throughout the life. The latter process of cutting cut back connections and strengthening the frequently used ones is called *synaptic pruning*. In the *frontal cortex*, which is the area for planning actions, decision making, controlling emotions, selecting and inhibiting responses, the synaptogenesis happens later and also the synaptic pruning takes longer and does not reach adult level until at least the age of 18. Several studies shows what happens if there is a lack of appropriate stimulations during a sensitive period. “However, appropriate input need not be in any way sophisticated. Instead it tends to be basic and general, and is readily available in normal environments. The presence of patterned and colored visual stimuli, sounds, and objects to touch and manipulate, for example, is ample stimulation for the developing sensory cortices of the human brain” (Blakemore & Frith, 2005, p. 26). This does not mean that ‘the richer the environment the better’, instead “it might be more accurate to say that a ‘normal’ environment leads to more synaptic connections than a deprived environment. It is unlikely that children brought up in any ‘normal’ child-oriented environment could be deprived of sensory input. The research does, however, suggest that there is a threshold of environmental richness below which a deprived environment could harm a baby’s brain” (Blakemore & Frith, 2005, p. 33).

Blakemore and Frith believe that the research evidence does not support a *selective* educational focus on the early years of children using hothousing. But they are here writing about children in rich countries. One could argue that children growing up in very impoverished environments need additional stimulation than what they get in their homes. Blakemore and Frith (2005) states that little is known whether specific experiences are required for developing nonsensory skills and the relevant brain areas for arithmetic, but there are evidence that several sensitive periods exists for the development of language. “One day, no doubt, research findings will illuminate what relevance sensitive periods have for skills and capacities that depend on formal education” (Blakemore & Frith, 2005, p. 31).

Some neural systems are highly neuroplastic and can be modified to either advance or be more vulnerable (Neville & Bavelier, 1999; Stevens & Neville, 2006). For instance children from very “talkative” families have a language that grows much more than children from moderately talkative families and non-talkative families (Hart & Risley, 1995). One might here conclude that changing the lives of poor families is a long process and we need to begin very early. Stevens asks (2008) if an early intervention can serve as a buffer and protect the vulnerable neural systems. The participating children all lived at or below poverty level. She particularly reports on a parent training model that helped the children on risk factors such as stress, language, and behavioral control/emotional regulation. They conclude that following a short-term parent training (eight weekly two-hour sessions) parents changed their behavior, and parents reported reduced stress, and the children's language, cognition, and attention had improved. This training was not done specifically for mathematics learning, however, we would argue that the areas of language, cognition, and attention are pre-conditions for learning mathematics more efficiently.

Implications on pre-primary teaching

Early intervention and pre-school programs ensuring stimulation should be initiated for poor students in developing countries. Globally in 2004, 1/3 of pre-primary age students were enrolled in pre-primary education and the global pre-primary GER (Gross Enrollment Rates) has risen by 33.4% since 1995 (World Bank, 2008). There is room for improvement; programs should not only focus on providing a stimulating environment, but should also train parents to for instance improve the students' attention. Here we would also like to point to the importance of providing good nutrition since malnutrition affects brain development and learning efficiency (Abadzi, 2006). It should therefore be considered that the parent training includes nutrition advice and that the pre-school programs in some cases provide school meals.

6. Cognitive Skills used in Primary Mathematics

In this section we focus on the impact of working memory, procedural versus conceptual understanding, metacognition as well as some general advice.

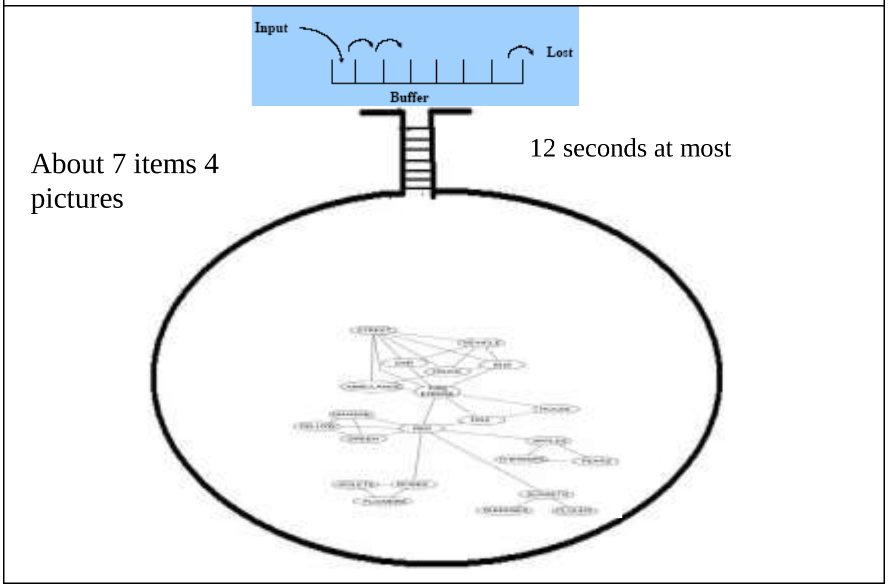
Short-term (or working) Memory

Following Bligh (2000), memories are connected through a central processor called 'central executive' which together with two 'slave systems' make the 'working memory'. "One of the slave systems is an articulatory loop and phonological store … Using sub vocal speech, this can extent the length of the verbal auditory short term memory from two to three or four seconds. … The other slave system is a comparable one for visual information … a 'visuospatial sketch pad'. Clearly, a lecture with any visual display uses both systems" (p. 27).

When various tasks are practiced enough, items are chunked together, so they pass as one from working memories. The association and quick connection of items is the basis for automaticity. When various items or tasks are recalled automatically, the mind makes little effort to bring them up. Thus, attention and the limited working memory span can be directed towards problem solving (Flor & Dooley, 1998, p. 168; Blakemore & Frith, 2005, pp. 29-31).

Sousa (2008) described working memory as a "work table, a place of limited capacity where we can build, take apart, or rework ideas for eventual disposal or storage somewhere else" (p. 51). Sousa's (2008) metaphor illustrates that working memory has an "upper limit". The amount of information a person can learn at any given time is relatively small. If one tries to put too much information in short term memory, some of the information will "fall off the table" and never make it to long term memory. In order for information to pass from short term to long term memory, one has to go through the process of rehearsal and memorization of small amounts of information at a time. According to Sousa (2008), that the upper limit for working memory increases as a child ages. Children who are two can remember two "chunks" of information. Preadolescents can handle a number of five chunks of information.

Figure 4: The structure of short- and long-term memory



Source: Abadzi 2006

Working memory is especially critical to mathematics learning because mathematics lessons place frequent demands on working memory Cathercole et al. (2006). Students must remember intermediate products of calculations in order to solve problems. Good working memory has therefore been shown to be correlated with successful mathematics learning. Conversely, those who have poor mathematics skills have problems with working memory (Passolunghi et al., 2007). Passolunghi et al. (2007) found that working memory and the ability to count are the two most salient precursors of early mathematics learning. Similarly, Bull et al. (2008) showed that working memory predicted mathematics achievement in the students' first three years of elementary school.

Goswami (2008, p. 282) writes that "small amounts of training can lead to rapid improvement in the strategic use of rehearsal, with accompanying improvements in recall". This is also the case for children down to 7 years. 4-year olds, however, appear to not show improvement in their memory. Furthermore, Goswami (2008) states: "Organizational mnemonic strategies, such as sorting required grocery items into related groups and using this clustering to aid recall, show a similar developmental pattern to rehearsal" (p. 283). In fact, using multiple strategies makes children able to recall even more information (p. 285). Also, "mothers who use an elaborative conversational style tend to have children who have more organized and detailed memories" (p. 293). The latter, we argue, again points to the importance of pre-primary education for poor students in developing countries.

Procedural versus conceptual understanding and the value of automaticity

Automatic manipulation is necessary. Skemp (1987) distinguishes between 'routine manipulations' and 'problem-solving activity'. He states that unless the routine manipulations can be done 'with minimal attention', it is not possible for the students to successfully concentrate on the difficulties. However, to what extent should students practice mainly procedures to the point of automaticity vs. develop conceptual understanding?

The question of conceptual versus procedural understanding has been discussed for decades. Waves seem to have gone back and forth between what is best. "A major conflict in educational theory and practice is between 'formalism' in education and what is known as 'progressive education' movement. The formalists tend to emphasize subject matter. … Progressivism is a protest against formalism. It places emphasis upon the interests and desires of the individual, upon freedom, and upon the child rather than upon any particular subject matter" (Titus, 1946, p. 19). Kilpatrick also states: "Why is it that so many intelligent, well-trained, well-intentioned teachers put such a premium on developing students' skill in the routines of arithmetic and algebra despite decades of advice to the contrary from so-called experts? What is it that teachers know that others do not?" (Kilpatrick, 1988). The mathematics psychologist Skemp (1987, pp. 158-159) argues for three advantages with skill based instrumental understanding ('rules without reason') – such as "to divide by a fraction you turn it upside down and multiply":

1. Within its own context instrumental mathematics is usually easier to understand,
2. So the rewards are more immediate, and more apparent,
3. Just because less knowledge is involved, one can often get the right answer more quickly and reliably by instrumental thinking than relational.

Relational understanding/thinking occurs when one has built up a conceptual structure (schema) of mathematics and therefore both know *what* to do and *why* when one solves a mathematical problem. Skemp (1987, p. 160) states that there are several reasons why a teacher teaches for instrumental understanding such as:

1. Relational understanding would take too long to achieve, and the students only need to use a particular technique.
2. Relational understanding of a specific topic is too difficult, but the student needs it for examination reasons.
3. A skill is needed for use in another subject before it can be understood relationally with the schemas presently available to the students.

Treffers et al. (2001, p. 147) distinguish between 'algorithm calculations' (traditional algorithms) and 'column calculation' using a 'splitting strategy' where interim results are calculated. They have a balanced view of the pros and cons of the two types of calculations: "Learning the calculation algorithm requires at least one hundred class hours. … An early introduction to algorithm calculation and an extensive sequel form a major obstacle to the development of mental arithmetic with handy, varied calculations; it also hampers estimation. … Column calculation promotes mental arithmetic and estimation partly due to the calculation structure from large to small … Column calculation links up naturally with the informal approaches used by children … Children can learn the algorithm-based addition procedure … in about five lessons after they have become familiar with column addition" (p. 149).

Along with the discussion on procedurally vs. conceptually oriented instruction, there is much debate worldwide on whether students should explore and discover concepts or whether they should learn them explicitly. Advanced and very bright students may excel at discovery but for the average students the research suggests that explicit instruction is more efficient (Kirschner et al. 2006). This issue is discussed more extensively in section 8.8, in relation to curricular reforms.

The US. National Council of Teachers of Mathematics panel (2008) has attempted to resolve these differences. Members noted that failure of American students to master fractions is the greatest obstacle to learning algebra. To achieve these, fewer topics can be covered in greater depth. Also Schmidt (2002) characterizes the US curricula as "A Mile Wide, an Inch Deep" (p. 2). To prepare students for algebra and advanced mathematics, the curriculum must simultaneously develop the following goals:

- conceptual understanding
- computational fluency
  - Automaticity of basic skills
- problem-solving skills
  - reading fluency in order to read problems
- proficiency in operations of whole numbers and fractions, negative numbers

Implications of research of working memory and automacity for teaching

Abadzi (2006) pointed out that best instructional practice takes into account the limits of working memory. This is why automatizing the operations that are used for intermediate calculations (multiplication tables, frequent subtractions) is an important and early function to acquire in mathematics.

There is no doubt that the development of routines and automaticity is of vital importance in the learning process. Without this, the students are not able to fully concentrate on complex problem solving, mathematical reasoning, and modelling. Without practice, the neural connections will not be strengthened; hence pruning will not take place. Learning appropriate algorithms is also of vital importance.

When a small, age-appropriate number of mathematical ideas are presented at a time, more learning occurs and less instructional time is wasted. Furthermore, it is important to teach the students various mnemonic strategies and also to use visual aids.

Metacognition

'Metacognition' can be understood either as knowledge about or regulation of cognition (Schoenfeld, 1992, p. 334; Goswami, 2008, pp. 295-333). Knowledge about cognition means to have relatively stable information about one's own cognitive processes. This knowledge develops with age and "performance on many tasks is positively correlated with the degree of one's metaknowledge" (Schoenfeld, 1985, p. 138). Metacognition, understood as regulation of cognition, includes the planning before beginning to solve a problem and the monitoring and assessing "on-line" during problem-solving and learning (Schoenfeld, 1992, p. 355). The presence of this has a positive impact on intellectual performance and the absence a strong negative effect. It also includes 'metamemory' (knowledge about memory) (Goswami, 2008).

A study showed that at least by the age of 9, children are able to assess the relative usefulness of the rehearing, categorizing (by semantic category), looking and naming strategies for remembering (Goswami, 2008, p. 298). Eight-year olds had some ability to self-regulate their memory behavior hence allocate study time, and the pattern was stronger for 10 and 12 year old students. In contrast, six-year olds lacked adequate metamemorial knowledge that would help them allocated more time on difficult problems. However, some researchers have studied the judgments-of-learning in children aged 6, 8, and 10 years, which showed that in some circumstances, even kindergarten children display accurate self-monitoring. There is also a positive correlation between metamemory and memory performance (Goswami, 2008, pp. 299-304).

Studies on 7-years old investigated the possible effects of metacognitive support on children's analogical reasoning. The results showed that 98% of the children in the metacognitive group solved the problem using a 'taught' strategy while only 38% of the children in the control did the same. "The children were 'learning-to-learn', learning to use analogy even though they were never instructed explicitly in how the problems were alike" (Goswami, 2008, p. 326). In terms of conditional reasoning (deductive), metacognition also plays a role even for children. "Children with good Metacognitive skills are 'good information processors'. They can use Metacognitive strategies to improve their memories … monitor their performance … and they can evaluate their memory behavior (Goswami, 2008, p. 332).

The concept of metacognition has been integrated into Singapore's primary mathematics curriculum. It is identified as a key component of mathematical problem solving (Singapore Ministry of Education, 2006). The teacher can help their students find their best way of working if the teacher (Chinn & Ashcroft, 2007, p. 287):

- Begins each lesson with an overall picture of its contents, using both oral and visual stimuli.
- Thoroughly explains the logic behind each method.
- Offers alternative methods.
- Puts the work into a familiar context, or relates it to the students' own experiences and existing knowledge.

Similarly, Singapore's Ministry of Education (2006) gives the following list of activities, which may be used to develop students' metacognition:

- Expose students to general problem solving skills, thinking skills and heuristics, and how these skills can be applied to solve problems.
- Encourage students to think aloud the strategies and methods they use to solve particular problems.
- Provide students with problems that require planning (before solving) and evaluation (after solving).
- Encourage students to seek alternative ways of solving the same problem and to check the appropriateness and reasonableness of the answer.

- Allow students to discuss how to solve a particular problem and to explain the different methods that they use for solving the problem. (p. 9)

Implications on research on metacognition for teaching

The concept of helping students monitor their own capacity to learn and to engage in mathematical thinking has received little attention thus far. Teachers are typically not taught to enhance this skill. However, developing metacognition and metamemory even in young children seems to be a way to enhance the learning and reasoning skills for the children being studies. There is no reason to doubt that this will also be the case for all children in developing countries.

7. Specific Teaching and Learning Practices

In this section we will provide research results and teaching recommendations for a selected number of specific teaching and learning practices such as when to start formal teaching, lesson structure, homework, workbooks, use of concrete materials, individual versus group work – also in terms of ability levels - use of computers and calculators, and learning in another language.

Some instructional advice from educational research

There is much cognitive research on making learning efficient that is applicable for mathematics instruction. The following sections focus on various aspects of this research. Below are some examples:

- Telling students and showing at the same time allows processing in more than one cognitive network (dual thinking mode). For example, Participants told how to solve a problem solved 13%, those shown solved 28%, those told and shown solved 40% (Reed and Bolstad 1991).

- **The importance of analogies**. Noticing that a past solution is relevant and mapping the elements of that solution onto the current problem are of paramount importance. A study analyzed how analogies -- a reasoning practice that involves connecting two concepts, often a better-known concept to a less familiar one -- are used in the United States, Hong Kong and Japan (Richland et al. 2007).. They are known to be helpful for learning mathematical concepts, **but only if teachers use enough imagery and gestures that direct students' attention to the analogous relations**. These strategies, or cognitive supports, are necessary to ensure that students notice and understand the analogies. U.S. teachers incorporate analogies into their lessons as often as teachers in Hong Kong and Japan, but they less frequently utilize spatial supports, mental and visual imagery, and gestures that encourage active reasoning. Less cognitive support may result in students retaining less information, learning in a less conceptual way, or misunderstanding the analogies and learning something different altogether.
Similarly pertinent research is related to following solved examples.

- **Spacing practice**. Distributive practice has particularly important implications in math. Students who review weeks or day after first studying the material are more likely to remember it in the long run. Thus, math textbooks should not only cover a single topic per unit but also introduce exercises from earlier units (Pashler et al. 2007).

Relying on research conducted by Gogtay et al. (2004) and Sousa (2008) on brain development and math, we will present the following findings and general recommendations for teaching pre-adolescent students mathematics:

Research findings	**Possible implications for learning mathematics**
The volume of the brain continues to grow until puberty.	Children can tackle problems of increasing difficulty as they move through the intermediate grades. There is no 'learning pause' in the intermediate grades.
At puberty, when the brain is nearly at its full adult size, gray matter volume begins to decrease because unneeded and unhealthy neurons are pruned away.	By sixth grade, creative problem solving should start becoming easier, include more options, and show greater sophistication of thought.
Parts of the brain associated with basic functions mature early. Motor and sensory functions (taste, smell, and vision) mature first, followed by areas involved in spatial orientation, speech and language development, and attention (upper and lower parietal lobes).	Primary grade children may have some difficulty solving complex visual-spatial problems. Boys may do better than girls at these types of challenges in the early grades, but the gap narrows in the intermediate grades. Multimodality approach likely to be successful.
Later to mature are those areas involved in executive functions (creativity, problem solving, reflection, analysis), attention, and motor coordination (frontal lobes).	These skills are just emerging in the intermediate grades, so problems with multiple approaches and answers are a challenge, but doable.
Most areas of the temporal lobes mature early. These areas are involved mainly in auditory processing. Maturing last is a small section of the temporal lobe involved in the integration of memory, audiovisual association, and recognition of objects.	Because the auditory areas are rapidly maturing, reading problems aloud is helpful. Three-dimensional object rotation and manipulation would be difficult for intermediate grade students.

Use of Math Workbooks

Workbooks have the advantage of helping structure learning tasks, can easily be incorporated as part of classroom routines (Truelove, Holaway-Johnson, Leslie & Smith, 2007) and can help students keep their work organized and in one place, which helps them review material at home. Moreover, a study by Tan, Lane and Coustere (1997) showed that elementary school students in the Philippines who had their own mathematics workbook had higher achievement scores than students that did not.

If possible, we recommend the use of one-use workbooks for each student. It is possible for work to be completed through slates, notebooks, and textbooks. However, based on the available information, this would not be the optimal situation in terms of organization and achievement scores. If workbooks are not available, we believe that the next best option would be for the teacher to require students to keep an organized notebook.

Use of Concrete Instructional Materials

Manipulatives

According to Goldin (2008) children do not always learn what teachers think they learn while working with the concrete materials. The reason is that a decontextualised representation is not the same as a genuine abstraction. Furthermore there is a risk of the *Jourdain*-effect which is the giving of a scientific name to a trivial activity (Brousseau, 1997, pp. 25-26). It is when we describe the productions of our students in mathematical terms, which presuppose an elaborate conceptual activity, while having no evidence of such an activity. During the New Math reforms in 1960s and 1970s, children received many kinds of 'manipulatives' to play with such as toys and blocks. When children sorted the toys, their activities were described using set theory terms such as 'finding the intersection of two sets'. "Example: The student asked to perform rather strange manipulations with jars of yoghurt or coloured pictures is told, 'You have just discovered a *Klein group*'" (p. 26).

However, in a review of the literature, Clements (1999) noted that students who learn mathematics using manipulative materials usually outperform students who do not use manipulatives. (This may be partly because moving manipulables may create groups that then facilitate students' understanding.) This result holds across grade levels and mathematics topic area. Manipulatives also improved student attitudes towards mathematics and increased their performance on problem solving tasks. Clements (1999), however, cautioned that the use of manipulatives does not automatically guarantee success; *how* manipulatives are used matters. Teachers must be aware of whether or not students are properly reflecting on their actions as they use the manipulatives and not just using them in a rote fashion. Stein and Bovalino (2001) found that successful classroom implementation of manipulatives were related to:

a) Professional development: Teachers had extensive training in the use of manipulatives.
b) Design of original lessons: Teachers designed their own lessons—this is an indicator of the amount of time a teacher spends preparing the lesson.
c) Strategic classroom planning: Teachers assigned students to groups and arranged manipulatives after considering how their students would use the manipulatives within their particular group situation.

There has been considerable debate over whether manipulatives help students understand fundamental mathematical concepts, such as the concept of equality (McNeil & Jarvin, 2007). McNeil and Jarvin (2007) recommend that given the mixed results of the effectiveness of manipulatives, teachers should:

a) Minimize the use of manipulatives that are highly concrete and rich in perceptual detail (McNeil et al., in press)
b) Minimize the use of toys or other manipulatives that are familiar to the students, because they may not serve in the expected roles.

The use of these types of manipulatives might engage the students only at the level of entertainment and at the expense of students developing a deep understanding of a mathematical concept. One could surmise that manipulatives rich in perceptual detail add another 'chunk' of information into a very limited working memory space when the child is learning a new concept. Further, McNeil and Jarvin (2007) recommend that when manipulatives are used in the classrooms, teachers should take the time to help students make a connection between their intuitive sense of mathematics and the formal language of mathematics.

Similarly, Schwartz and Varma (in press) showed that some manipulatives are more effective in facilitating the transfer of mathematical knowledge than others. In a study of students learning to add simple fractions with tile pieces and pie pieces, the students who learned with tile pieces were much more successful at giving correct symbolic answers to simple fraction addition problems than were students who learned with pie manipulatives. Schartz and Varma (in press) explained that the poor performance of the students who learned with the pie-shaped manipulatives relied on the part-whole structure of pies and could not transfer the concept of part-whole to other situations; specifically, they could always derive a part from a whole, but not create a different whole from several parts because of the shape of the pie pieces. The children who learned with tiles could create a whole from parts or parts from a whole.

Zhou and Peverly (2005) describe specific teaching practices that make the use of an established manipulative—the abacus—a successful mediator between concrete mathematics and abstract mathematics. The success of the abacus as a manipulative tool is rightly attributed to the pedagogical context in China. Firstly, teacher preparation in China emphasizes the appropriate use of teaching materials such as manipulatives. Secondly, when manipulatives are used, "direct instruction is frequently used because the Chinese believe that young children are not capable of making conceptual connections between concrete and abstract mathematical representations" (p. 261).

Implications of research on the use of manipulatives for teaching

The evidence on the usefulness of concrete material is ambiguous and at least some teachers have to have extensive training in order for it to work. When used properly, there however seems to be little doubt that it does help some students. For instance the abacus is successful because it is used by teachers who understand it as part of a whole system of teaching mathematics. Equally we can expect that such material also could be useful for poor students in developing countries. However, given that teachers do not always have a lot of training and that such concrete material is expensive, we will not recommend investing in concrete material – as a top priority. However, whenever teaching preparation is rather well, we will suggest that the prospecting teacher (or in-service teachers) learn about such methods since some of the material might be produced by the teachers, or students or parents themselves from material (wood, …) that they already have.

Software and Games to Improve Performance

Teaching relationships among the symbols and the concepts is effective, as was shown by the work of Brian Butterworth (2008) with Australian aboriginal students. Several researchers have developed software for this purpose. This includes Rightstart and Number Race. Older games like Snakes and Ladders have also proved useful. There is also some evidence on the utility of chess, introduced in particular in the US state of Idaho.

Ramani and Siegler (2008) found that playing linear board games with consecutively numbered, linearly arranged spaces provide opportunities to learn about the relation between numerals and their sizes. They enhanced numerical knowledge among low-income children along the following domains: numerical magnitude comparison, number line estimation, counting, and numeral identification. Specifically, board games enhanced numerical knowledge among low-income children along the following domains: numerical magnitude comparison, number line estimation, counting, and numeral identification. Results of earlier research by Siegler and Ramani (2006) on number line estimation can be readily adapted to the classroom. Siegler and Ramani (2006) showed that number line estimation among low-income, preschool aged children improved after playing with numbered board games during a two week period. During this two week period, they played the board game four times, at fifteen minutes a session.

Seidig (2007) pointed out that game-based computer learning makes mathematics fun, which can in turn engage students and motivate them to study mathematics. Kliman (1999) has three criteria to evaluate if a software game is good for educational use: the mathematical content, software suitability, potential to engage students. Kliman also states that students can start appreciate the mathematics as something useful and interesting in itself if exploring mathematics is a central ‘play’ of a computer game. He also states that in many programs, this is not the case but mathematics is something that the children need to do before the ‘real’ computer game begins – i.e. something the children have to do before the real fun begins.

Rosas et al. (2003) investigated the effects on learning, motivation, and classroom dynamic by introducing video games looking like Nintendo’s Gameboy into the class room. The study involved 1274 students in grades 1 and 2. The students used on average 30 hours over a 3 months period. The learning mechanism behind the use of video and computer games as pedagogical tool is what is called ‘incidental learning’ (Rosas et al., 2003, p. 77) which is the learning of structures of knowledge in the absence of explicit presentation of knowledge. According to Rosas et al. (2003. p. 75), what make games effective are: clear objectives, adequate complexity, speed, includes instructions during the game, independent on physical laws, and ”holding power” – they catch the player’s attention and make them build up their own world. The research showed a significant difference between experimental group and the external control group, but not a significant difference between the experimental and internal control groups. “In the case of this study, the Hawthorn’s effect occurred in a systematic and explicit manner: teachers of the internal control groups were aware of the experiment, and therefore made special efforts to accomplish an adequate performance of their students, sometimes trying to ‘compete’ with achievements in the EG” (Rosas et al., 2003, p. 89). Another interesting result was that even though the computers took time away from normal teaching, the students still learnt the same mathematics as the students in the control group.

Computers and calculators

Kulik and Kulik (1991) analyzed 254 studies on the use of computer-based instructions (CBI) on all levels from kindergarten to adult students. This showed that CBI programs raise the examination scores by 0.30 standard deviations. However, the effects were larger in published studies compared to unpublished studies such as dissertations and technical documents. Particularly one could see that the average effect size were significantly higher in studies using CBI in a short time (4 weeks or less) than for longer studies – regardless of levels of teaching (Kulik & Kulik, 1991, p. 84). Kulik and Kulik discusses the reason for the why the effects are significantly stronger in studies where students have been exposed to “treatment” in a shorter time. They mention a novelty effect, or Hawthorne effect. Furthermore, “As the treatment grows familiar, it loses its potency. But it is also possible that shorter experiments produce stronger results because short experiments are more tightly controlled experiments” (Kulik & Kulik, 1991, p. 89).

A study by Vanderbilt University (2008) showed that it is not a problem to use calculators in elementary classes if the students already have some basic skills and facts. The study was done on third graders. “These findings suggest that it is important children first learn how to calculate answers on their own, but after that initial phase, using calculators is a fine thing to do, even for basic multiplication facts” (Vanderbilt University, 2008). It is the level of a student’s knowledge of mathematics that was the determining factor in indicating if a calculator was hindering the students’ learning. The study also showed that if students did not know many “multiplication facts, generating the answers on their own, without a calculator, was important and helped their performance on subsequent tests. … But for students who already knew some multiplication facts, it didn’t matter - using a calculator to practice neither helped nor harmed them” (Vanderbilt University, 2008). In fact, for students who were not good at multiplying, using the calculator had a negative impact on their performance. However, students who used calculators to practice more problems had fewer errors.

Kulik (2003) reported that instructional technology often improves teaching programs in mathematics. Educational software known as integrated learning systems (ILS) “provide sequential instruction for students over several grades while keeping extensive records of student progress. Most ILS programs use tutorial instruction as a basic teaching methodology, and most provide instruction in the basic skill areas of reading and mathematics” (p. 52). Sixteen controlled studies of integrated learning systems show that mathematics achievement scores that were higher among groups of students taught with ILS than among students who were taught without ILS. In seven of these studies, the scores were statistically and practically significant. Recent studies have shown that instructional technology such as calculators and software programs have a positive effect on mathematics achievement in elementary students (Polly, 2008; Seidig, 2008; Suh & Moyer, 2007). Furthermore, Moor and Brink (2001, p. 209) states that the calculator can be used as “a didactical enrichment for mental arithmetic, estimation, column calculation and algorithms, gaining insight into the position and value system, and the basic operations”.

Implications of research on computers and calculators for teaching

There seems to be little doubt that the correct use of computers and calculators aid the learning of mathematics. We might further argue that when using such technologies in the class room, the students improve their technology competencies which are something that is of general use in the workforce. However, as with other teaching methods, computers and calculators are not cure-all solutions. For instance, it is very important to not use calculators before the students have some basic skills. We would recommend that even though it can be expensive, investing in computer technology aids learning and improves the students’ general technology competencies. The availability of instructional technology is more important than concrete materials such as manipulatives. Instructional technology makes it possible to use the wide range of virtual manipulatives that are available.[4]

[4] National Library of Virtual Manipulatives: http://matti.usu.edu/nlvm/nav/vlibrary.html
National Council of Teachers of Mathematics, electronic examples: http://standards.nctm.org/document/eexamples/
On virtual manipulatives: http://otec.uoregon.edu/virtual_manipulatives.htm

Individual versus group work – and ability levels

In this section we merely focus on the possible merits of group work. This is in no way to indicate that individual work is not useful. However, in the section we investigate research talking about if there are *also* merits of group work.

Heterogeneous Grouping of Students

Studies by Linchevsk and Kutscher (1998) and Burris and Levin (2006) showed that mixed ability classrooms have been shown to positively affect students' interest in taking more advanced mathematics and students' mathematics achievement. Similarly Wood and Frid (2005) demonstrated that multi-age settings can give rise to supportive learning environments for elementary school students provided that the teacher prepares thoroughly and meets the varied developmental needs of the students. Winebrenner (2001) argues for placing gifted students in their own cooperative learning group while the rest of the class placed in heterogeneous groups. However, if the task is focused on critical thinking and problem-based learning, placing gifted students in heterogeneous groups might be the best since students can benefit from a variety of view points. In general, if a teacher can answer yes to all the following three questions, heterogeneous cooperative learning groups are probably the most appropriate, otherwise gifted students should be placed by themselves (Winebrenner, 2001, p. 174):

1. Does the task require input from different types of learning style and different perspectives?
2. Is the subject matter new for all students?
3. Is it likely that the gifted students will be engaged in real learning rather than continuous tutoring?

Cooperative Learning

In a literature review on cooperative learning, Slavin et al. (2003) stated that there is consensus among researchers in this area that group learning results in positive outcomes across grades, ability levels, and subject areas. Kutnick et al. (2008) showed that students who participated in an experimental group designed to promote working in groups out-performed students who participated in the control group with regard to their academic achievement and willingness to work with other students. Sloane (2007) discussed the feasibility of small mathematics study groups. These groups can be used for practice, skill development, and exploration of mathematics as "a supplement to teacher-led instruction."

Fuchs et al. (2002) described an experimentally successful peer assisted learning program (PALS) and gave a protocol detailing how the groups are to be structured, the expectations for each groups and specific teacher prompts. Teachers reported that the PALS program was effective and feasible. The study results showed that PALS had a positive effect on first grade mathematics achievement along the achievement continuum (students with disabilities to high achieving students). In the discussion, the researchers pointed out that high achieving students profit from PALS because of the opportunity to explain concepts to lower-achieving peers.

Souvignier and Kronenberger (2007) found that in using the jigsaw method, a cooperative learning technique that requires all students to take turns being an expert in teaching material to their peers, requires planning. In their sample of 3rd graders, they deduce that the children should be trained in asking questions and giving appropriate verbal prompts in order for successful implementation of this method. Structuring the groups is important to insuring its successful implementation.

Other studies show that the question of whether to use group work or not is related to whether the problems the students work with are new and how much memory is needed to solve those problems. McNeese (2001) states that groups of two people solved the hardest part of a problem more often and more quickly than students working by themselves, however, problems relying on routine solutions and memory are best done by individuals alone.

Implications of research on group work for teaching

Group work is not a cure-all teaching method, but used right, it can help produce good learning outcome. However, in order for this to happen the teacher should take into consideration when there is a benefit of homogenous versus heterogeneous grouping as well as how to place gifted students and students with learning difficulties such as discussed above. However, all the above is said in context of usual first-world classroom size. In some developing countries, the usual class size is perhaps 60 wherefore such methods might not be as useful.

Homework

Recent studies of homework show that it is positively associated with academic achievement (Cooper, Robinson & Patall, 2006). Furthermore, homework is a valuable motivator and teaches values that help prepare students for the workplace such as responsibility, efficiency, and time management (Bempechat, 2004; Corno & Xu, 2004). Homework has been shown to be positively related to mathematics achievement among 4th 5th and 6th graders (Pezek, Berry & Renno 2002). However, the frequency and length of homework has been shown to have a differential effect. In a study of seventh grade German students, Trautwein, Koller, Schmitz& Baumert (2002) found that homework that is *frequently assigned* has positive effect on mathematics achievement. In contrast, homework that is l*engthy* has a slightly negative effect on mathematics achievement. Also assigning more homework tends to have a larger and more significant impact on mathematics test scores for high and low achievers, and it is less effective for average achievers (Henderson and Eren 2008).

No information could be found with regard to the *optimum* amounts of mathematics homework per school night for elementary school students. Reynolds and Mujjs (1999) recommended that homework assignments should be:

1. Assigned on a regular basis at the end of each mathematics class.
2. Should involve about 15 minutes of work to be done at home.
3. Should include 1 or 2 review problems (Reynolds & Muijs, 1999, pp. 277-278).

Spacing Assignments: Importance of Distributive Practice

Research from brain research and cognition has established the importance of practice on learning and achievement. Research from Rohrer and Taylor (2006) give specific recommendations based on two recent experiments of university level mathematics students. The purpose of these experiments was to evaluate the effects of three different practice strategies. One strategy involved practice problems of one type spaced over several assignments, or distributed practice, another involved practice problems completed in one assignment, which is also known as 'massed practice'. The last strategy examined by Rohrer and Taylor (2006) is called overlearning, which refers to students continuing to practice a skill that they have already mastered. The results of their study showed that distributed practice resulted in long term retention of concepts. Furthermore, distributed practice was more effective than overlearning strategies in successful long term retention of concepts.

Rohrer and Taylor (2006) pointed out that most textbooks structure their problem sets according to 'massed practice' and 'overlearning strategies'. They proposed the alternative structure:

> Fortunately, there is an alternative format that minimizes overlearning and massed practice while emphasizing distributed practice, and it does not require an increase in either the number of practice sets or the number of problems per practice set. With this distributed practice format, each lesson is followed by the usual number of practice problems, but only a few of these problems relate to the immediately preceding lesson. Additional problems of the same type might also appear once or twice in each of the next dozen assignments and once again after every fifth or tenth assignment thereafter. In brief, the number of practice problems relating to a given topic is no greater than that of typical mathematics textbooks, but the temporal distribution of these problems is increased dramatically (p. 1218).

Issues of language use in math instruction

Research in the last two decades showed a clear advantage to learning mathematics in a student's native language (Adetula, 1985; Adetula 1990; Bernardo, 1999). For instance Adetula (1990) investigated the effect of presenting arithmetic word problems in the students' native language or in English to Nigerian students. His findings indicated that the students performed better when the word problems were presented in their native language. More recent scholarly work has produced more distinctions among the areas of second language learning in mathematics, such as a) degree of fluency in the second language b) bilingualism c) the role native language plays in mathematics problem-solving d) cognitive demands of learning mathematics in a second language.

Degree of fluency in the second language and bilingualism

In a study that explored the differential item functioning (DIF) on the National Assessment of Educational Progress (NAEP) between a group of fourth grade second-language English learners and native English speakers, Mahoney (2008) found that as a whole, the test items performed the same among native English speakers and second language English learners. Mahoney (2008) cautioned that when interpreting the results of this study, one should take into account that level of English-language proficiency was unknown for the group of second language

English learners who participated in the study. In general, degree of fluency should be accounted for in research studies and in teaching practice.

Related to the issue of fluency is bilingualism. Studies have shown that bilingual students outperform their monolingual peers in mathematics achievement tests (Clarkson, 1992; UNESCO, 2007). The key to the success of bilingual programs, however, is a curricular approach to bilingual education:

> "In Mali, [*Pédagogie Convergente*] involves not only a change in language of instruction; it is a bilingual curriculum with specific educational objectives, teaching and learning methods and materials. The student-centred, project-focused pedagogy has contributed significantly to improved learning outcomes in Mali's bilingual primary schools (UNESCO, 2007, p. 14)"

Role of native language in mathematics problem solving and the position of English

Two studies show promising results for situations in which quality bilingual instruction is not possible. Clarkson (2006) explained that language switching among bilingual Vietnamese/English students is based on group context and affective preference. Also, these students tended to switch between English and Vietnamese in the earlier grades and then choose English as the primary language for studying mathematics in the later grades.

The finding that language use is context specific in the mathematics classroom was also found by Barwell (2005). In a study of Year 5 students (ages 9-10), Barwell (2005) found that when working on arithmetic word problems "having English as an additional language is never explicitly relevant to participants' discussions. At no stage do any of the participants directly bring in issues of bilingualism, of difficulty with using English or of language being a problem… This observation suggests that, as Moschkovich (1999) suggests for teachers, where attention is maintained on the mathematics, language issues need not be problematic" (p. 345). Also, Jyotsna Vaid found that the language with which one is first *taught math* is the language most bilingual people use to solve mathematics problems (Texas A & M, 2001).

But there might be more at stake than "just" what language is best for learning. In 1996, Malawi introduced a major reform in her school language policy. The government directed that all students in grades 1 to 4 should, with immediate effect, learn in their mother tongue. However, Kamwendo (2003) writes that in Malawi there is a rising "appetite for English in an environment in which the language is more or less synonymous with education itself". This means that since English is a world-dominant language, the argument is that the students might as well get used to English as soon as possible.

We can also see such development in for instance the Philippines. Due to the decline of English, mathematics, and science proficiency, the president of the Filipinos issued in 2003 an Executive Order (EO 210) to strengthen the use of English as the language of instruction in the Philippine educational system. Since the 1987 Constitution, for purposes of communication and instruction, the official languages of the Philippines are Filipino and English. The EO 210 furthermore states that there is a "need to develop the aptitude, competence and proficiency of our students in the English language to maintain and improve their competitive edge in emerging and fast-growing local and international industries, particularly in the area of Information and

Communications Technology" (Manila Times, 2008). Hence, in order to maintain economic competitiveness, English might need to be a language with which the students are very familiar.

Implications for teaching

Regardless of whether students who learn mathematics in a second language are successful at mathematics or highly skilled at their second language, this situation presents additional cognitive demands on students that teachers must be prepared to face. Kasule and Mapolelo (2005) place these challenges within the following typology that includes a continuum of *cognitively undemanding communication,* to *cognitively demanding communication*.

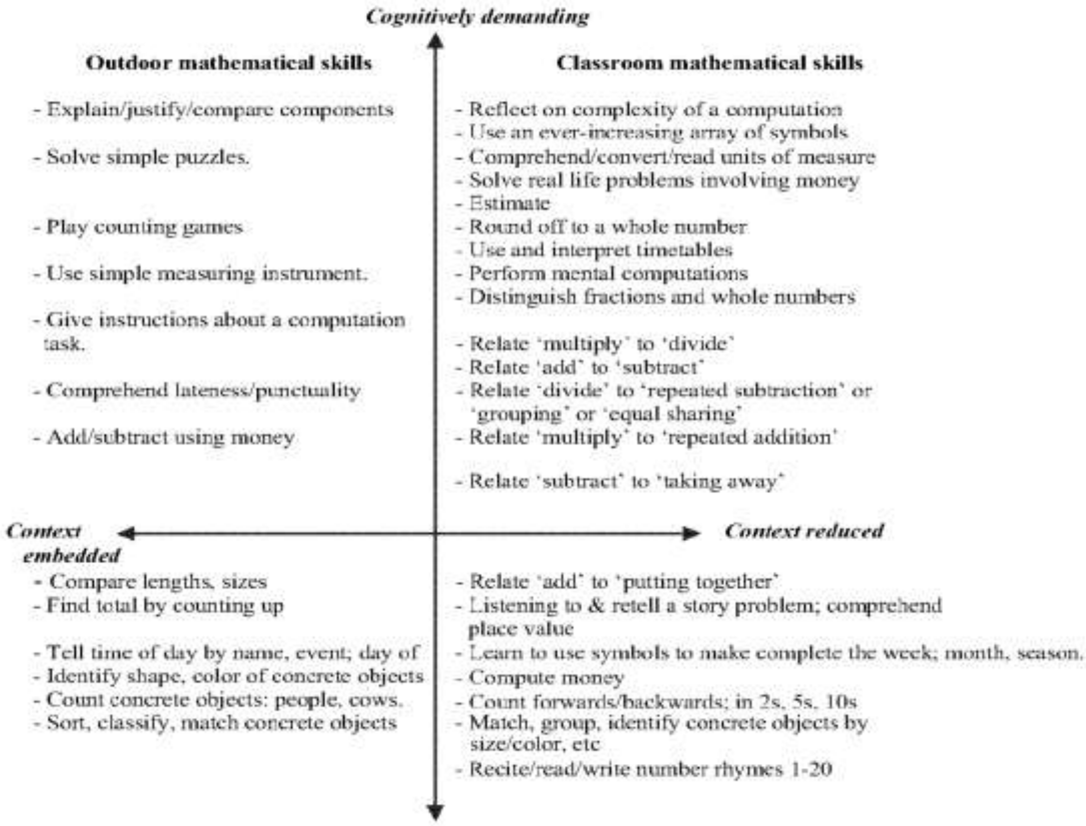


Fig. 1. The cognitive demands of language in elementary mathematics classrooms.

Campbell et al. (2007) outline the following questions that could be used to reflect on what kinds of support second language learners could use in a mathematics classroom:

- How experienced are students with mathematics concepts and procedures?
- How experienced are the students with concepts from other content areas, such as science and social studies that are required?
- What mathematical processes are needed and how experienced are the students at using them? What cognitive processing skills are needed?
- Do the students' prior experiences include the development of mathematical language and the development of the reflective and command functions of natural language in the learning of mathematics?
- Does the language used in the problem statement or instruction correspond to the level of English language development of ESL students?

- Are there words that have specialized meanings in mathematics that have different meanings in natural [everyday] language?
- What knowledge of cultural or life experiences is needed to understand the problem statement? What connections need to be made between the mathematics of the classroom and student experience?

There seems little doubt that "learning-wise" it is of benefit to teach in the native language. However, this requires that there are trained teachers to do that. Also since English is so world-dominant, we might argue that better learn it early and be fluent – perhaps have actual bilingual schools such as in Mali.

8. Curricular and Teacher Training Issues

Recommended Lesson Structure and Hours of Instruction

Regrettably, we could not find many recent research articles studying the impact of mathematics lesson structures on mathematics learning. However, Reynolds and Muijs (1999) cited the Missouri Mathematics Effectiveness Project implemented in the 1970s and 1980s, which reported promising results. Based on this project, the following general lesson structure is recommended for primary mathematics lessons:

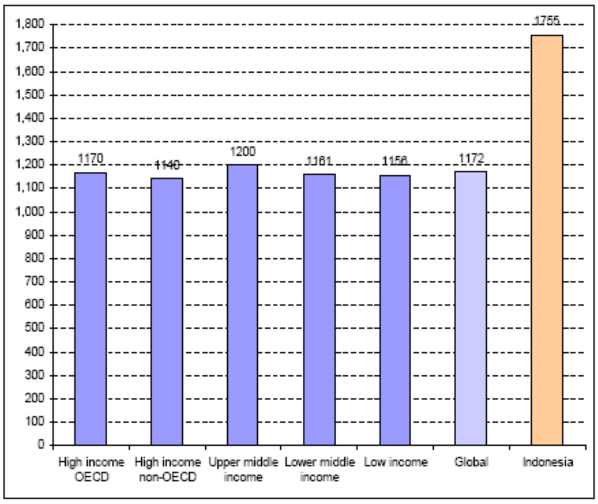
(a) Daily Review (approx. 10 minutes)
  1. Review concepts and skills associated with previous day's homework.
  2. Collect and deal with homework assignments.
  3. Ask several mental computation exercises.

(b) Development (approx. 20 minutes) (introducing new concepts, developing understanding)
  1. Briefly focus on prerequisite skills and concepts.
  2. Focus on meaning and promote student understanding by lively explanations, demonstrations etc.
  3. Assess student competence.
     a. Using process and product questions (active interaction).
     b. Using controlled practice.
  4. Repeat and elaborate on the meaning portion as necessary

(c) Individual Work (approx. 15 minutes)
  1. Provide uninterrupted successful practice.
  2. Momentum - keep the ball rolling - get everyone involved, then sustain involvement.
  3. Alerting - let students know their work will be checked at the end of each period.
  4. Accountability - check the student's work.

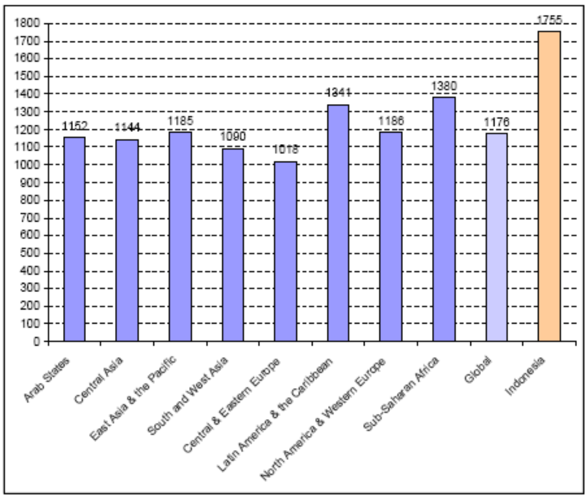
In terms of number of hours recommended, a study by Amadio and Truong (2006) states that globally for grades 1-9, countries allocate about 1,172 hours to mathematics instruction. They state: "an amount that seems fairly stable across income levels—ranging from a maximum of 1,200 hours in the case of upper middle-income countries to a minimum of 1,140 hours in high-income non-OECD countries. In Indonesia, the total number of hours to be spent on mathematics is considerably higher than the apparent pattern" (p. 3).

Figure 5. Median instructional time (in accumulated number of hours) allocated to mathematics over the first nine years of schooling, by income level



(Total number of cases = 110). All figures are medians, excepting data for Indonesia which refer to the intended number of instructional hours.

Figure 6. Median instructional time (in accumulated number of hours) allocated to mathematics over the first nine years of schooling, by EFA region



(Total number of cases = 113). All figures are medians, excepting data for Indonesia which refer to the intended number of instructional hours.

What might be concluded here is that it is not the exact number of hours which constitute evidence of quality teaching, more *what happens in those hours*. However, given the previously mentioned challenges with regard to teaching in developing countries such as language and stimulation issues, we would recommend to aim at the higher end of the number of hours of instruction recommended.

Recommendations for Curriculum Adoption and Benchmarks

Based on the research mentioned in Annex B, we recommend adopting a curriculum with benchmarks that is aligned with that of the A+ and Singapore curricula, because of the empirical evidence showing success in using these curricula. We also recommend using caution in implementing progressive programs in developing poor countries, based on the findings mentioned previously.

Across all the countries included in this report, the benchmarks provide for some flexibility in when the students should learn various skills. This flexibility is most easily seen in the UK curriculum, which specifies several levels of achievement targets. Specifically, the UK curriculum states that students at the same key stage, which is over several years, work at various levels. In the UK curriculum, there is no evidence to suggest a belief that all students follow *one route* of learning. Another consistent theme of primary mathematics curriculum is coverage of the following topics: *numbers, geometry, statistics, and algebraic reasoning.*

Comments on testing

In order to measure how well the students achieve the various benchmarks, it is worth noting that a publication from the OECD (2004) states that *formative assessment* is a very effective way to not only measure the students learning but also an important tool in order to improve the students learning. Formative assessment is a part of the teaching process. It is supposed to be incorporated during the teaching in order to provide information about the need to adjust the teaching and/or learning while the teaching is happening. This is different from summative assessments, which measure at a given time what students know and how much they know. Many countries such as Norway have developed such formative tests.

An example of a formative test is a diagnostic test, which gives the students problems to solve; the students are unable to answer these problems correctly if they do not understand the prerequisite concepts. Below is an example of a problem used to diagnose misunderstandings of the decimal system for students grades 4, 6, and 8 (Brekke, 2002, pp. 6-7):

Put a ring around the smallest number: 0.625 0.25 0.3753 0.125 0.5

A test revealed the following patterns of answer:

	Grade 4	*Grade 6*	*Grade 8*
0.125 (correct)	16	55	79
0.5	64	26	7
0.3753	8	13	10
0.25	8	4	3
0.625	1	1	1

Such answer patterns can give the teacher knowledge of not only the numbers of correct answers but also the erroneous reasoning that underlies these answers. The teacher can use this knowledge of students' false understandings during teaching.

Another indicator of performance is the use of fingers. A study by Jordan et al. (2008) showed that the use of fingers is a potential indicator of poor performance. There is a correlation between finger use and accuracy. It decreases gradually, from 0.60 in kindergarten to -0.15 at the end of second grade. Low-income children showed linear growth in frequency of finger use, middle-income children slowed down by second grade. Girls and boys showed similar growth patterns in frequency and accuracy whereas boys used their fingers less often than girls and were more accurate. [5]

We could not find sufficient research based information with regard to specific critical variables to consider when designing efficient mathematics assessments. Instead, general principles with regard to designing mathematics assessments have been published. For example, the Mathematical Sciences Education Board (MSEB) said that mathematics assessments should be evaluated against the educational principles of *content, learning* and *equity* (MSEB, 1993). The content principle is a basic validity question: to what extent does the assessment reflect the mathematics content that is most important for the students to learn? The benchmarks mentioned in the previous sections help the test designer meet this principle.

The learning principle asks the test designer to consider if the assessment leads to improved teaching and learning. In other words, does the assessment increase student and

[5] There are socioeconomic implications in testing (Gilmore et al. (2007)"More affluent kids tested in the laboratory did better than their less well off peers tested in their classrooms, the group reports. The reason for the difference could be the testing environment, says Spelke, who adds that the important point is that kids from diverse backgrounds all showed the ability". Thus, students in developing countries may have the same ability of learning mathematics as students in the developed world.

teacher expectations of performance? Does an increased performance, in turn, lead to improved learning?

Finally, the equity principle consists of several components:

> Several aspects of the [equity] principle require examination and evaluation. The first aspect involves the usual issues associated with equity of assessment: traditional questions of fairness and of comparability across groups, scores, tasks, and situations. The second aspect involves questions of whether students have the opportunity to learn important mathematics (whether they have been taught the important mathematics being assessed). The third aspect is newer and is associated with pedagogy that requires that all students find assessment tasks accessible if the tasks are to have the needed positive impact on their learning (MSEB, 1993, p. 129).

Comments on teacher training

In terms of teacher education, we do not want to give a comprehensive recommendation to all aspects within teacher education for developing countries. However, we will provide a few general suggestions that we believe are valuable and which would help point future efforts to improve teacher education in the right direction.

First, content. Schollar (2008, p. 16) refers to a report of "the President's Education Initiative (PEI)[6] summarizing the findings of 35 research studies commented that: "The most definite point of convergence across the PEI studies is the conclusion that teachers' conceptual knowledge of the subjects they are teaching is a fundamental constraint on the quality of teaching and learning activities, and consequently, on the quality of learning outcomes". A well grounded knowledge of mathematics is vital. But how much is then enough? Different countries have different requirements here not just in terms of the actual amount of mathematics but also what mathematics is taught. Sometimes the perception is that for teaching primary level children, not much mathematics is needed, it is more important to "love children". We do not agree. We believe that primary school teachers should have, ideally, a solid grounding of mathematics content. Singapore is one country that imposes strict admissions exams for entrance into the teacher training program.

Second, we would like to point to the concept of Pedagogical Concept Knowledge (PCK) introduced by Shulman (1986) which is "(1) was a subcategory of content knowledge; (2) is topic specific; and (3) included two further subcategories: knowledge of representations and or learning difficulties and strategies for overcoming them" (Hashweh, 2005, p. 275). We find that it is important that the (prospecting) teachers not only get a huge content knowledge base *and* a pedagogical/psychological knowledge base – these two do not integrate automatically in the teacher's mind. To address this, the concept of PCK describes the specific type of knowledge that a teacher needs to have in order to teach primary mathematics successfully.

[6] Taylor, N. and Vinjevold, P. (Eds.) "Getting Learning Right". (JET, 1999)

For countries in which there are severe teaching shortages or budgetary constraints, it might not be realistic to ensure that teaching institutions prepare the needed numbers of qualified primary mathematics teachers. To address this problem, the Commonwealth of Virginia, in the United States has piloted a program of Elementary School Mathematics Specialists to provide ongoing professional development in mathematics for elementary school teachers (Haver, 2008). Mathematics Specialists are “based in an elementary school in order to support the professional growth of teachers and promote excellent mathematics instruction and student learning. Mathematics Specialists are responsible for strengthening classroom teachers’ understanding of mathematics content and helping teachers develop more effective mathematics teaching practices. Typically they collaborate with individual teachers through co-planning, co-teaching, and coaching” (Haver, 2008, p. 1). The specific duties of the mathematics specialist are to:

- Collaborate with individual teachers through co-planning, co-teaching, and coaching;
- Assist administrative and instructional staff in interpreting data and designing approaches to improve student achievement and instruction;
- Ensure that the school curriculum is aligned with state and national standards and their school division’s mathematics curriculum;
- Promote teachers’ delivery and understanding of the school curriculum through collaborative long-range and short-range planning;
- Facilitate teachers’ use of successful, research-based instructional strategies, including differentiated instruction for diverse learners such as those with limited English proficiency or disabilities;
- Work with parent/guardians and community leaders to foster continuing home/school/community partnerships focused on students’ learning of mathematics; and
- Collaborate with administrators to provide leadership and vision for a school-wide mathematics program (Haver, 2008, p. 6).

In an evaluation of the mathematics specialist program in Virginia, Haver (2008) found that when compared with elementary schools that did not have a Mathematics Specialists, students in schools with Mathematics Specialists scored significantly higher on the Virginia Standards of Learning in Mathematics in grades 3, 4, and 5. Within each of these grade levels, student performance in schools with mathematics specialists was higher in each of the following areas of mathematics: Number, Number Sense, Computation, Estimation, Measurement, Geometry, Probability and Statistics, and Patterns, Functions, and Algebra.

To sum, mathematics content and pedagogical content knowledge must be strengthened in teacher education programs, professional development efforts after teaching, or both.

9. Learning Disabilities and Mathematics: Dyscalculia

Though humans have an innate number sense, some people struggle with a brain disorder that interferes with normal mathematical thinking (Shalev, 2004). Dyscalculia is a learning disability that exists among students in both developed and developing countries. Geary (2006) stated that “Between 3 and 8% of school-aged children will show evidence of dyscalculia” (p. 1). Shalev and Gross-Tur (1993) defined developmental dyscalculia as a primary cognitive disorder of childhood that affects normal acquisition of arithmetic skills. Landerl et al. (2004) concluded that dyscalculia is not a result of reading or language deficits, but on problems with basic numerical processing. Finally, dyscalculia is not unitary but includes several subtypes with different characteristics. For example, Kadosh and Walsh (2007) stated that dyscalculia can include deficits in different mathematical abilities such as:

- automatic processing of numerical information
- the efficiency of making associations between symbolic meaning and quantity — the figure '7'and 'seven-ness'
- retrieving arithmetical facts
- executing efficient calculation procedures (For example, it is normal for six year old children to count with their fingers in order to solve arithmetical problems, but adopting the same strategy at the ten years of age is a sign of age-inadequate arithmetic skills).

Symptoms of dyscalculia include: a slowness in giving answers relative to the typical student, reliance on counting tangible objects, difficulties with mathematical terms and language, poor memory of mathematical facts, difficulties in comprehending sequences of numbers, difficulties with position and spatial organization, and over-reliance on rote learning and imitation (Hannell, 2005). Wilson and Dehaene (2007) gave possible neurocognitive core reasons for deficits in mathematical reasoning. One possible root cause of dyscalculia is a deficit in number sense, which is related to impairment in the horizontal intra-parietal sulcus (HIPS) area. Another possible cause of dyscalculia is the lack of sufficient connections between non-symbolic and symbolic representations of number. Finally, Wilson and Dehaene (2007) proposed three possible causes for various subtypes of dyscalculia:

- deficits in verbal symbolic representation, which would hinder learning basic mathematical facts
- executive dysfunction, which would also inhibit the ability to retrieve mathematical facts or
- spatial attention, which would impair inherent number sense.

General recommendations for teachers of students with dyscalculia

Based on what is currently known about dyscalculia, Wilson and Dehaene (2007) stated that teaching methods based on enhancing number sense should be effective. They also pointed out the possibility of developing interventions based on verbal memory training, visuospatial, attention or executive attention training. Shalev (2004) gave the following general recommendations for teachers, advising that teachers can:

- Teach repetitive additions, for example, using 10, that is, 4 + 10 = 14, 14 + 10 = 24, 24 + 10 = 34
- Focus on the neuropsychologic problem underlying the dyscalculia, whether it is perceptual and visual-spatial or verbal and auditory. This approach stresses verbalization of arithmetic concepts, procedures, and operations.
- Use instructional technology remediation programs such as MASTER (Mathematics Strategy Training for Educational Remediation) developed for teaching multiplication and division. "The efficacy of this program suggests that children with dyscalculia can learn arithmetic when provided with number concepts and problem-solving strategies."

- Encourage practice: Children who understand number concepts but have difficulty in computation can be overwhelmed by the procedures needed to solve an arithmetic exercise. Therefore, enhancing automatic recall for number facts by drill and practice can also be helpful.
- Use representational systems to facilitate learning, an example of which is a thermometer for the concept of a number line.
- Encourage students to verbalize their perception of the arithmetic procedure while the teacher provides feedback.
- Use aids such as "Pocket calculator[s]. These are helpful when impaired memorization of number facts impedes the ability to correctly complete an arithmetic problem."

Wilson et al. (2006) also reports of the software 'The Number Race' as being effective for disabled children aged 5-8, but which may also be useful for normal preschool children (see earlier section).

Chinn and Ashcroft (2007) also have a number of suggestions. One is that the curriculum should be characterized by being relaxed, welcoming, empathetic, and have a low-stress atmosphere. This is in order for the students to feel confident asking questions. Furthermore, the beginning of the course should be relatively easy to give success experiences to students who might have been labeled a failure. It is also important that the work is not too easy and perceived as patronizing. Furthermore, the structure of the course should be characterized by a spiral with a small pitch that allows regular revisits of the same topics to give the students the opportunity of essential over-learning since dyslexic need continual reminders and memory refreshers. Topics also need to change frequently to promote and sustain interest. For numeracy, the order of topics suggested is the following, but it should be regarded as a continuum instead of separate skills (pp. 282-283):

- Sorting and classifying
- Counting with whole numbers and using them to measure and draw
- Adding in whole numbers
- Subtracting in whole numbers
- Multiplying in whole numbers
- Dividing in whole numbers
- Understanding about parts of whole numbers
- The four operations for money
- The four operations for decimals
- The four operations for fractions.

For geometry and algebra, topics such as perimeter, area, equations, angle-sum, and graphs should be introduced only when the required numeracy level has been acquired. They recommend introducing algebra early and in the form of simple formulae when concluding pieces of fully understood work. The students should also be encouraged to do mental calculations: "They should not be expected to invent them by themselves, but any method that they have already adopted should be welcomed" (p. 286). In terms of the latter, we would therefore argue that discovery learning does not seem to be appropriate for this type of students.

Recommended interventions for specific mathematics learning problems

Dowker (2001) gave intervention recommendations based on specific problem areas in mathematics. The following table outlines these recommendations.

Problem area	**Intervention**
Principles and procedures related to basic **counting**;	Children practice counting and answering cardinality and order-irrelevance questions about very small numbers of counters (up to four) and are then given further practice with increasingly large sets. For repeated addition by one and repeated subtraction by one, children are given practice in observing and predicting the results of such repeated additions and subtractions with counters (up to 20). They are then given verbal 'number after' and 'number before' problems: 'What is the number before eight?', 'What is the number after 14?', etc.
Use of **written arithmetical symbolism**;	Children practice reading and writing numbers. Children with difficulties in reading or writing two-digit numbers (tens and units) are given practice in sorting objects into groups of ten, and recording them as '20', '30', etc. They are then given such sorting and recording tasks where there are extra units as well as the groups of ten.
Use of **place value** in arithmetic;	Children are shown the addition of tens to units and the addition of tens to tens in several different forms: (i) written numerals; (ii) number line or number block; (iii) hands and fingers in pictures; (iv) ten pence pieces and pennies; (v) any apparatus with which the child is familiar. The fact that these give the same answers should be emphasized. Children whose difficulties are more specific to the use of place value in arithmetic are given practice with arithmetical patterns such as: 20 + 10; 20 + 11; 20 + 12 etc. and are encouraged to use apparatus when necessary.
Understanding and solution of **word problems**;	Children are given addition and subtraction word problems, which are discussed with them: "What are the numbers that we have to work with?" "What do we have to do with the numbers?" "Do you think that we have to do an adding sum or a taking-away sum?" "Do you think that John has more sweets or fewer sweets than he used to have?" etc. They are encouraged to use counters to represent the operations in the word problems, as well as writing the sums numerically.
Translation between **concrete, verbal and numerical Formats**	Children are shown the same problems in different forms, and shown that they give the same results. They are also encouraged to represent word problems and concrete problems by numerical sums, and to represent numerical problems and word problems by concrete objects.

Though dyscalculia is a problem that requires intensive intervention, the good news is that the techniques suggested do not require expensive materials and can be viewed as extensions of good teaching practice. For example, finger use, which does not require 'extra material' is potentially useful as a complementary method to help primary grades students with dyscalculia establish and internalize mental number representations and learn to calculate (Kaufman, 2008).

10. Concluding comments

This paper has endeavored to collect and integrate various researches with the aim of answering important questions about mathematics teaching and learning. Our goal was to use sound research studies to help formulate recommendations about best practices for primary school students in developing countries. However, this research field is still evolving and a lot is still not known about the learning of mathematics. We anticipate that the growing understanding of neuroscience and cognition will, in time, give very valuable information about mathematics learning that will result in more specific guidelines to give to teachers.

We would also like to note that even though research does enable us to formulate various recommendations, children are very different and we cannot expect, ever, to find *the* solution what will fit all. Children are different, in part, because they are subject to vastly different classroom processes; many things happen in the classroom – it is a complex entity in which affective and motivation factors also play a large role. However, humans are also not that different – we tend to vary and develop within the same frames wherefore we can expect to at least be able to give answers that point in the right direction.

Future directions include the following: (1) We would suggest to have further "state-of-the-art" investigations based on research that has already been conducted. (2) We would also suggest having a critical discussion of progressive teaching philosophy. A lot of research seems to contradict it, but still it seems to be *the* ideology taught and claimed. (3) Most of all, research in the learning of mathematics mostly seem to take place in the first world countries and on first world children. In order to have better suggestions to how to teach primary children in the developing world, more research need to be done there.

Annex A

How to Improve Math Instruction and Performance for the Poor?
Consultant Terms of Reference for a Literature Review

The consultant is to review research on mathematics education, particularly for primary education. The literature review constitutes a first phase of research on math performance for poor students and will focus on primary education. Subsequent stages may focus on greater levels of detail and on higher levels of schooling, as well as a review of the older math-by-radio programs. The consultant who will carry out this study will collect and integrate pertinent research to arrive at grade-level achievement indicators, curricular recommendations, and efficient means of math instruction (particularly for the poor). Specifically the following issues would be covered:

- A brief, layman-oriented overview of how the brain produces mathematics, particularly among younger children: the triple code, development of a number line, relationships with spatial orientation, interactions with working memory, etc. At this time, are there specific implications from the neurocognitive research regarding math instruction, particularly in the early grades? If so, what are they?

- Math achievement problems commonly encountered among the poor, a condition sometimes called dyscalculia. What is known about these (mainly from developed countries), what are specific features, what solutions have been found to improve dyscalculic students' math achievement? How can these inform improvements in instruction for students of low-income schools at the international level?

- Given research outcomes and results of review panels, (e.g. National Mathematics Advisory Panel) what should students know at the end of each grade in grades K-6? What specific benchmarks should be expected at the end of grades K, 1, 2, 3, 4, 5, 6? For example, the National Panel on mathematics has determined that by the end of grade 6, students should be fluent on all four operations and on the use of fractions.

- What general principles does research suggest for improving math instructional efficiency in grades K-6? Some examples are below, and the consultant will add according to expertise:

    - Should teachers continue to use concrete objects as aids given negative research outcomes?
    - How should instruction reconcile the need for conceptualization with the need for memorization of multiplication tables and automaticity acquisition?
    - Should instruction focus on applications and procedures (poor TIMSS results in the US) rather than proofs and explanation of procedures?
    - Should students be allowed to struggle for solutions or should they be put through proofs and solved exercises in expectations of learning the general processes?
    - What is known about the results of individual vs. group problem solving? i.e. when should schools be encouraged to let the students discuss problem solutions and when should students work on their own?

- What research exists regarding the role of computers in providing necessary instruction and practice? (Several countries start to use inexpensive computers for children that must be programmed correctly.)

- Given research, what specific applications should be preferred in schools that have limited instructional materials and possibly poorly trained teachers? How to remediate most efficiently the math skills of students who have fallen behind?

- What is known about teaching math in a language other than a child's mother tongue? A brief literature review would update current knowledge (see the relevant chapter in Abadzi 2006);

- What are the curricular implications for obtaining the desired outcomes in every grade? Ideally how many weekly hours of math should be taught, how many hours of homework? Should materials include one-use workbooks, or could students reasonably expect to complete the work through textbooks, slates, and notebooks?

- How could achievement of the various benchmarks be tested most efficiently? Are there perhaps critical variables that can lead to brief and easily administered tests (an example from reading is oral reading speed, which highly correlates with comprehension.)

- What general principles does research suggest for improving the training of primary school math teachers with limited education?

- What organizations and individuals could provide support specifically for math achievement issues for the poor in low-income countries? Any information would be appreciated.

- Overall, what recommendations should the staff of the World Bank give to governments to help improve math learning outcomes for the poor?

The product would be a relatively succinct paper with annexes as needed, and a bibliography list. It could also be accompanied by pictures, videos, and computer programs if available. The consultant would produce an outline for review after approximately seven days of work for discussion and reformulation of questions as necessary based on the literature review.

Annex B

Benchmarks for K-6 in numbers and geometry

In this section we will discuss the benchmarks of The Netherlands, Denmark, the UK, USA, and Singapore primary education mathematics. However, the term 'primary' varies. In The Netherlands and the UK, children begin primary school at age 5. In the USA, students usually start kindergarten at the age of 5, and grade 1 at the age of 6. Also in Singapore, primary education normally commence when the child reaches the age of 6 years. In Denmark, compulsory schooling begins at the age of 7. These differences in when children begin formal schooling are one of the reasons why we chose to look at the benchmarks of these countries. But it is an important difference to keep in mind when comparing the benchmarks. Another reason for the choice of countries is to get an overview over how countries with different teaching traditions have chosen to structure their primary school in mathematics.

The Netherlands

In The Netherlands, the objectives of primary education (attainment targets) prescribes how teaching should be structured and organised, but the content of teaching itself as well as the teaching methods are not prescribed. Primary education is 8 years and finishes at the age of 12. The attainment targets for primary school do in broad terms define the core curriculum to ensure that students are prepared for secondary school (Dutch Eurydice Unit, 2007). For numbers and geometry, the attainment targets are as follows (Dutch Ministry of Education, Culture and Science, 2008):

Core objectives
Numbers and calculations
- The pupils learn to understand the general structure and interrelationship of quantities, whole numbers, decimal numbers, percentages, and proportions, and to use these to do arithmetic in practical situations.
- The pupils learn to quickly carry out the basic calculations in their heads using whole numbers, at least to 100, whereby adding and subtracting up to 20 and the multiplication tables are known by heart.
- The pupils learn to count and calculate by estimation.
- The pupils learn clever ways to add, subtract, multiply and divide.
- The pupils learn to add, subtract, multiply and divide on paper, according to more or less contracted standard procedures.
- The pupils learn to use the calculator with insight.

Measuring and geometry
- The pupils learn to solve simple geometrical problems.
- The pupils learn to measure and calculate using units and measurements, such as time, money, length, circumference, surface area, volume, weight, speed, and temperature.

As can be seen, these objectives are stated in very overall terms and are not grade or age specific. It is also not very precise except that it is specifically stated that multiplications tables should be learnt by heart. Also the use of text books is a commercial activity but to help schools choose, the Ministry published guides that compare the quality of all teaching material. Hence, looking into such material, one might be able to "backwardly" deduce what had been the age level targets since the correlation between what textbooks cover and what teachers teach is 0.95 (Schmidt et al. 2002). However, this is not the scope of this paper. Instead we want to show the Dutch example to illustrate what one country decided to do.

The UK

The UK is the other country where children start primary education at the age of five. Here, a National Curriculum with four key stage goals has been in existence since the 1988 Education Reform Act to accompany the first introduction of the National Curriculum. The National Curriculum is organized as follows (Qualifications and Curriculum Authority, 2008):

Key stage 1: ages 5-7 (grades 1-2)
Key stage 2: ages 7-11 (grades 3-6)
Key stage 3: ages 11-14 (grades 7-9)
Key stage 4: ages 14-16 (grades 10-11)

Besides these key stages, the curriculum also divides up in 8 attainment levels as well as an exceptional level. The students are supposed to work at as follows:

Levels 1-3 in key stage 1 and attain Level 2 at the end of the key stage
Levels 2-5 in key stage 2 and attain Level 4 at the end of the key stage
Levels 3-7 in key stage 3 and attain Level 5/6 at the end of the key stage

Children who are high-achieving may be awarded a Level 8 or Exceptional Performance. Below we have inserted the attainment targets for number & algebra and shape, space & measures. Since this paper deals with primary education, we have only referred to the Levels 1-5:

Attainment target 2: Ma2 Number and algebra

Level 1
Pupils count, order, add and subtract numbers when solving problems involving up to 10 objects. They read and write the numbers involved.

Level 2
Pupils count sets of objects reliably, and use mental recall of addition and subtraction facts to 10. They begin to understand the place value of each digit in a number and use this to order numbers up to 100. They choose the appropriate operation when solving addition and subtraction problems. They use the knowledge that subtraction is the inverse of addition. They use mental calculation strategies to solve number problems involving money and measures. They recognize sequences of numbers, including odd and even numbers.

Level 3
Pupils show understanding of place value in numbers up to 1000 and use this to make approximations. They begin to use decimal notation and to recognize negative numbers, in contexts such as money and temperature. Pupils use mental recall of addition and subtraction facts to 20 in solving problems involving larger numbers. They add and subtract numbers with two digits mentally and numbers with three digits using written methods. They use mental recall of the 2, 3, 4, 5 and 10 multiplication tables and derive the associated division facts. They solve whole-number problems involving multiplication or division, including those that give rise to remainders. They use simple fractions that are several parts of a whole and recognize when two simple fractions are equivalent.

Level 4
Pupils use their understanding of place value to multiply and divide whole numbers by 10 or 100. In solving number problems, pupils use a range of mental methods of computation with the four operations, including mental recall of multiplication facts up to 10 x 10 and quick derivation of corresponding division facts. They use efficient written methods of addition and subtraction and of short multiplication and division. They add and subtract decimals to two places and order decimals to three places. In solving problems with or without a calculator, pupils check the reasonableness of their results by reference to their knowledge of the context or to the size of the numbers. They recognize approximate proportions of a whole and use simple fractions and percentages to describe these. Pupils recognize and describe number patterns, and relationships including multiple, factor and square. They begin to use simple formulae expressed in words. Pupils use and interpret coordinates in the first quadrant.

Level 5
Pupils use their understanding of place value to multiply and divide whole numbers and decimals by 10, 100 and 1000. They order, add and subtract negative numbers in context. They use all four operations with decimals to two places. They reduce a fraction to its simplest form by cancelling common factors and solve simple problems involving ratio and direct proportion. They calculate fractional or percentage parts of quantities and measurements, using a calculator where appropriate. Pupils understand and use an appropriate non-calculator method for solving problems that involve multiplying and dividing any three-digit number by any two-digit number. They check their solutions by applying inverse operations or estimating using approximations. They construct, express in symbolic form, and use simple formulae involving one or two operations. They use brackets appropriately. Pupils use and interpret coordinates in all four quadrants.

Attainment target 3: Ma3 Shape, space and measures

Level 1
When working with 2-D and 3-D shapes, pupils use everyday language to describe properties and positions. They measure and order objects using direct comparison, and order events.
Level 2
Pupils use mathematical names for common 3-D and 2-D shapes and describe their properties, including numbers of sides and corners. They distinguish between straight and turning movements, understand angle as a measurement of turn, and recognize right angles in turns. They begin to use everyday non-standard and standard units to measure length and mass.
Level 3
Pupils classify 3-D and 2-D shapes in various ways using mathematical properties such as reflective symmetry for 2-D shapes. They use non-standard units, standard metric units of length, capacity and mass and standard units of time, in a range of contexts.
Level 4
Pupils make 3-D mathematical models by linking given faces or edges, draw common 2-D shapes in different orientations on grids. They reflect simple shapes in a mirror line. They choose and use appropriate units and instruments, interpreting, with appropriate accuracy, numbers on a range of measuring instruments. They find perimeters of simple shapes and find areas by counting squares.
Level 5
When constructing models and when drawing or using shapes, pupils measure and draw angles to the nearest degree, and use language associated with angle. Pupils know the angle sum of a triangle and that of angles at a point. They identify all the symmetries of 2-D shapes. They know the rough metric equivalents of imperial units still in daily use and convert one metric unit to another. They make sensible estimates of a range of measures in relation to everyday situations. Pupils understand and use the formula for the area of a rectangle.

We see that these targets are very detailed and the structure of the levels allow for flexibility within each grade level.

USA

In the USA there are no national standards or national curriculum. Standards are determined by the individual states. There is considerable overlap in coverage, and some states adopt the standards and guidelines recommended by the National Council of Teachers of Mathematics (NCTM). NCTM is a national organization for mathematics teachers. NCTM first developed Standards for mathematics in 1989 and it is voluntary whether or not a state wants to adopting the NCTM Standards. However, according to NCTM (2008), most states do base their standards and benchmarks on the NCTM Standards. The Standards were revised in 2000, and are as follows for numbers and geometry:

Pre-K – 2: Numbers
All students should:
•count with understanding and recognize "how many" in sets of objects;
•use multiple models to develop initial understandings of place value and the base-ten number system;
•develop understanding of the relative position and magnitude of whole numbers and of ordinal and cardinal numbers and their connections;
•develop a sense of whole numbers and represent and use them in flexible ways, including relating, composing, and decomposing numbers;
•connect number words and numerals to the quantities they represent, using various physical models and representations;
•understand and represent commonly used fractions, such as 1/4, 1/3, and 1/2.

Grades 3 - 5: Numbers

All students should:
•understand the place-value structure of the base-ten number system and be able to represent and compare whole numbers and decimals;
•recognize equivalent representations for the same number and generate them by decomposing and composing numbers;
•develop understanding of fractions as parts of unit wholes, as parts of a collection, as locations on number lines, and as divisions of whole numbers;
•use models, benchmarks, and equivalent forms to judge the size of fractions;
•recognize and generate equivalent forms of commonly used fractions, decimals, and percents;
•explore numbers less than 0 by extending the number line and through familiar applications;
•describe classes of numbers according to characteristics such as the nature of their factors.

Pre-K – 2: Geometry
All students should:
•recognize, name, build, draw, compare, and sort two- and three-dimensional shapes;
•describe attributes and parts of two- and three-dimensional shapes;
•investigate and predict the results of putting together and taking apart two- and three-dimensional shapes.

Grades 3 - 5: Geometry
All students should:
•identify, compare, and analyze attributes of two- and three-dimensional shapes and develop vocabulary to describe the attributes;
•classify two- and three-dimensional shapes according to their properties and develop definitions of classes of shapes such as triangles and pyramids;
•investigate, describe, and reason about the results of subdividing, combining, and transforming shapes;
•explore congruence and similarity;
•make and test conjectures about geometric properties and relationships and develop logical arguments to justify conclusions.

A comparison with the UK system is not immediately possible. There are targets for the end of grade 2 in both the UK and USA, but the UK students begin a year earlier than the students in the USA. The UK Key stage 2 ends at grade 6, where the students are at the same age as the US students at grade 5, where the Standards for grades 3-5 ends. But at this time, the UK students have gone to school one more year than their contemporaries in the United States. Hence, a comparison of the targets has to be done with care.

Denmark

In Denmark, compulsory schooling begins at the age of 7 and is for 9 years. Years 9 years take place in a unit school (*Folkeskole*/People's School) and hence consists of what is typically called primary and lower secondary levels. The state/municipality schools share a common aim, standard requirements regarding each subject that should "be taught at the specific form levels, standard regulations concerning the so-called Common Objectives for the teaching in the individual subjects … However, it is the responsibility of the individual municipal boards to determine how the municipality's schools are to be organized in practice, within the framework established by law" (Danish Ministry of Education, 2008). The objectives are given as "Step Objectives" for the grades 3, 6, and 9 respectively, as well as "Final Objectives" for grades 9/10. Grade 10 is an optional grade in compulsory school. For matters of comparison, we have put the

Step Objectives around the areas of numbers, algebra, and geometry for grades 3 and 6 below (Danish Ministry of Education, 2006):[7]

Numbers and algebra

The students should by the end of grade 3 be able to:

- Know about the structure of the natural numbers, their order, counting-out rhyme and base 10 system
- Determine amount/number by using simple calculation in their head, counting material, calculator and written notes
- Know examples to practical problems that can be solved using addition and subtraction
- Work with preparatory multiplication and very simple division
- Know about examples of use of decimal numbers e.g. in relation to money, simple fractions such as 1 and 1/4.

The students should by the end of grade 6 be able to:

- Know about whole numbers, decimal numbers, and fractions
- Use experiences from daily life together with work in the school in order to build up number sense
- Know the order of numbers, number line, position system and the four operations
- Use mental calculation, estimations and written calculations
- Use calculators and computers in calculations
- Work with counting and examples on connections and rules within the four operations
- Know about examples of the use of variables, including their part in formulas, simple equations and functions
- Know about the concept of percentage, and connect it to daily life experiences
- Calculate with decimal numbers and use fractions in connections with percentage and concrete problems
- Work with “changes” and structures as they are part of number series, series of figures, and patterns
- Know about the coordination system, including the connection between number and drawing.

Geometry

The students should by the end of grade 3 be able to:

- Talk about daily things and pictures using geometrical language and base on forms, position and size
- Work with simple concrete models and reproduce traits from the reality of the drawing
- Investigate and describe patterns, including symmetry
- Work with simple measurement of distance, surface, space and weight
- Investigate and experiment within geometry *inter alia* using computers.

The students should by the end of grade 6 be able to:

- Use geometrical models and concepts in describing physical objects from daily life, including figures and patterns
- Investigate and describe simple figures drawn in the plane
- Know about basic geometrical concepts such as angles and parallelism
- Work with physical models and draw simple drawings of these
- Know about various cultures’ methods of indicating depth in pictures
- Investigate the particular drawing methods usefulness to describe form and distance
- Measure and calculate circumference, area, volume in concrete situations
- Draw, investigate and experiment with geometrical figures *inter alia* using computers.

Again, as stated above, due to the different age that the students begin primary school, a comparison of the Step Objectives with the UK Key Stages and NCTM Standards should be done with caution. For instance, the Danish grade 6 is not equivalent to the UK grade 6 since the UK students are two years younger. Neither the UK nor the USA has Key Stages of Standards for grade 3, but age-wise, the Danish grade 3 students would be equivalent to the US grade 2.

[7] The Step Objectives have been translated by the author Dahl Soendergaard since there is not an official English translation.

Lessons from TIMSS 1995 on A+ curricula

We would now like to turn to at study (Schmidt et al., 2002) on the results of the Third International Mathematics and Science Study (TIMSS) in 1995 where six countries statistically outperformed at least 35 other countries and were by this study called the “A+ countries”. These were: Singapore, Korea, Japan, Hong Kong, Belgium (Flemish-speaking), and the Czech Republic. The A+ countries’ curricula had a remarkably similar content. The study made comparison with the 21 participating US states whose curricula also between themselves were quite similar. The authors in the study of Schmidt et al. (2002) states that it is naïve to think that one can “lift something from one cultural context and expect it to work in another” (pp. 1-2). However, a study by Hook et al. (2007) implemented a “Quality curriculum” very much aligned with the common curricula of the A+ countries. Hook et al. (2007) refer to research stating that “it was the *content* of the curricula of the six leading math countries which was primarily responsible for their superior performance” (p. 130). The study of Hook et al. (2007) took place over five years and involved over 13,000 participants and showed that the performance of students having been taught using the Quality Curriculum outperformed the students who continued with the old curriculum ($0.003 < p < 0.015$). An overall picture of the A+ Curriculum can be seen below (Schmidt et al., 2002). What might be even more interesting is that also “school districts with a high percentage of economically disadvantaged and English learning immigrant students … were found to be statistically superior to similar (control) districts” (p. 143).

FIGURE 1

A+ Composite: Mathematics topics intended at each grade by at least two-thirds of A+ countries.

Note that topics are introduced and sustained in a coherent fashion, producing a clear upper-triangular structure.

TOPIC GRADE:	1	2	3	4	5	6	7	8
Whole Number Meaning	■	■	■	▒	▒			
Whole Number Operations	■	■	■	■	▒			
Measurement Units	□	■	■	■	■	■	▒	
Common Fractions			□	■	■	▒		
Equations & Formulas			□	▒	▒	▒	■	■
Data Representation & Analysis			□	□	▒	▒		□
2-D Geometry: Basics			□	▒	▒	▒	■	■
Polygons & Circles				▒	▒	▒	■	■
Perimeter, Area & Volume				▒	▒	▒	▒	□
Rounding & Significant Figures				▒	▒			
Estimating Computations				▒	▒	▒		
Properties of Whole Number Operations				□	▒			
Estimating Quantity & Size				□	□			
Decimal Fractions				▒	■	▒		
Relationship of Common & Decimal Fractions				▒	■	▒		
Properties of Common & Decimal Fractions					▒	▒		
Percentages					▒	▒		
Proportionality Concepts					▒	▒	▒	□
Proportionality Problems					▒	▒	■	■
2-D Coordinate Geometry					□	□	▒	▒
Geometry: Transformations						▒	▒	▒
Negative Numbers, Integers & Their Properties						□	▒	
Number Theory							▒	□
Exponents, Roots & Radicals							▒	▒
Exponents & Orders of Magnitude							□	□
Measurement Estimation & Errors							□	
Constructions w/ Straightedge & Compass							■	□
3-D Geometry							▒	■
Congruence & Similarity								■
Rational Numbers & Their Properties								□
Patterns, Relations & Functions								□
Slope & Trigonometry								□
Number of topics covered by at least 67% of the A+ countries	3	3	7	15	20	17	16	18
Number of additional topics intended by A+ countries to complete a typical curriculum at each grade level	2	6	5	1	1	3	6	3

□ – intended by 67% of the A+ countries ▒ – intended by 83% of the A+ countries ■ – intended by 100% of the A+ countries

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Singapore

Singapore primary mathematics curriculum is of special interest to this report since fourth and eighth grade students from this country have achieved top average scores and been at the first place in mathematics on TIMSS in 1995, 1999, and 2003 (MOE, 2000; 2004). It is also one of the 6 curricula that was part of the A+ curriculum discussed in the previous section. Ginsberg and Leinwand (2005) attribute this success to the following factors:

- Singapore has a uniform national mathematics curriculum and implements uniform assessments that measures student progress longitudinally.
- Singapore provides an alternative framework for students who struggle in mathematics.
- Textbooks in Singapore have an in-depth treatment of mathematical topics. Textbooks cover less material than standard U.S. mathematics textbooks but cover those topics in greater depth.
- A concrete-pictorial-abstract approach is used when presenting mathematical concepts in mathematics texts. “The Singapore textbook features instructional presentations in which a concept is first illustrated concretely, then pictorially, and finally abstractly. The approach

… tightly connects its concrete examples with student learning of mathematical ideas" (p. 6).

- Prospective primary school teachers must pass a "rigorous entrance exam to be accepted to education school, which they are paid to attend" (p. 9).

The latest primary mathematics standards adopted in 2006 by Singapore's Ministry of Education (MOE), is as follows:

OBJECTIVES OF THE PRIMARY MATHEMATICS CURRICULUM
PRIMARY 1 TO PRIMARY 4, PRIMARY 5 AND PRIMARY 6
The objectives of the primary mathematics program are to enable pupils to:
• Develop understanding of mathematical concepts:
    • Numerical
    • Geometrical
    • Statistical
    • Algebraic
• Recognise spatial relationships in two and three dimensions
• Recognise patterns and relationships in mathematics
• Use common systems of units
• Use mathematical language, symbols and diagrams to represent and communicate mathematical ideas
• Perform operations with: *whole numbers, fractions, and decimals*
• Use geometrical instruments
• Perform simple algebraic manipulation
• Use calculators
• Develop ability to perform mental calculation
• Develop ability to perform estimation
• Develop ability to check reasonableness of results
• Present and interpret information in written, graphical, diagrammatic and tabular forms
• Use mathematical concepts learnt to solve problems
• Use appropriate heuristics to solve problems
• Apply mathematics to everyday life problems
• Think logically and derive conclusions deductively
• Develop an inquiring mind through investigative activities
• Enjoy learning mathematics through a variety of activities

Specific recommendations for each subject area are quite detailed and focused; Ginsberg and Leinwand (2005) noted that this was a particular strength of Singapore's mathematics program. An example of this detail and focus is found for the specific objectives for Primary 1 addition and subtraction as follows:

Instructional content includes:
• concepts of addition and subtraction,
• use of the addition symbol (+) or subtraction symbol (−) to write a mathematical statement for a given situation,
• comparing two numbers within 20 to tell how much one number is greater (or smaller) than the other,
• recognizing the relationship between addition and subtraction,
• building up the addition bonds up to 9 + 9 and committing to memory,
• solving 1-step word problems involving addition and subtraction within 20,
• addition of more than two 1-digit numbers,
• addition and subtraction using formal algorithms.
• addition and subtraction within 100 involving
    ∗ a 2-digit number and ones,
    ∗ a 2-digit number and tens,
    ∗ two 2-digit numbers

One can see that these objectives are much more detailed than particularly the Dutch shown above, but also the others.

Whole curricular efforts: Traditional Versus Reform Based Mathematics

There is ongoing debate in the field of mathematics education with regard to 'reform- based' mathematics versus 'traditional' mathematics. In practice, the dividing line between traditional and reform based curricula is hard to determine. However, definitions of reform based programs as alternatives to traditional programs have been made. For example, the Core-Plus Mathematics Project offered the following taxonomy to help characterize their curricular materials:

	Traditional Approach	CPMP Approach
Instructional Focus	facts and procedures, with some applications and problem solving	conceptual understanding, problem solving, mathematics done in context, with requisite work on procedures and facts
Method	direct instruction, memorize and practice, with some projects and occasional group activities	active student engagement in inquiry and investigation, guided by probing teacher questions, with teacher-led introductions and summaries, and some direct instruction
Supplementary Material	problem solving, applications, thinking skills	additional skill practice

The Institute of Educational Sciences What Works Clearinghouse (2008) evaluated five elementary school curricula. Of the five, only one—*Everyday Mathematics*—was shown to have "potentially positive effects" on students' mathematics achievement. What is notable about the *Everyday Mathematics* curriculum is that it emphasizes a balance between computational and arithmetic skills, rigorous mathematics concepts, and meaningful problem solving (Everyday Mathematics, 2008). Having a balance between developing computation skills and conceptual skills has more potential to be inclusive than programs that do not have this balance.

Baxter et al. (2001) point out that students who have difficulty in mathematics are presented with "verbal and social challenges" in a classroom that places a high emphasis on student inquiry and investigation. Further, such programs place a high demand on teachers, who must learn reform-based pedagogies.

Other researchers (Webb, 2003) have investigated the impact of the 'Interactive Mathematics Program' (IMP), which is a text book series aligned with the 1989 Standards for high school students. It is a problem-based progressive curriculum that includes the fields of algebra, geometry, and trigonometry as well as some other topics not usually found in traditional high school curricula. Three studies collected data from students in nine public high schools from 1989-1997. One of the purposes of the studies was to investigate if being in an IMP-class increases student achievement. The result was that "students who had enrolled in *IMP* in Grade 9 generally performed comparably with students enrolled in the traditional *college-preparatory* mathematics course sequence on common measures on mathematics. … The only significant difference for any of the contrasts occurred at Brooks High School [synonymous], where students who started *IMP* in Grade 9 and took the SAT scored significantly higher on the mathematics section than did students who started in the traditional sequence in Grade 9 and took the test" (Webb, 2003, p. 385). One could argue that perhaps the *IMP* program works for students who are high achieving in mathematics. Such an argument might be supported by the work of Christian et al. (2001) who stated: “Direct sustained instructions can serve to reduce the large individual differences produced by other factors prior to school entry” (p. 326). This means that curricula for poor students might benefit much for from more “traditional” curricula.

It is important to keep in mind that this study was conducted in a first-world country. In South Africa, Schollar (2008, pp. 16-18) criticises constructivist theory of learning and states that: “In recent years a steadily growing body of both local and international research has questioned the nature of the curriculum itself and, in particular, the learning theories upon which it is based and the associated teaching practices it promotes. In short, this research is increasingly calling into question the effectiveness of constructivist ‘learner-centred’ OBE-based methodology in achieving high levels of performance, especially in mathematics” (p. 16). He refers to various studies showing that ‘constructivist’ ideas such as collaborative group work were not significant in relation to gaining high scores. Another critique of constructivism is provided by Matthews (2000, pp. 498-499): “There is an ‘Evidential Dilemma’ for constructivists: they wish to appeal to the nature of cognitive realities … and epistemological realities … to support their pedagogical, curricular and epistemological proposals, yet simultaneously maintain that such reality cannot be known and, for some, does not even exist. … Constructivists … create an *in principle* barrier between evidence and theory. This then leaves space for ideology, personal and group self interest, or just ‘feel-goodness’ to determine theory choice and educational policy”. Furthermore, Matthews writes that: “Learning theory is not epistemology: the mechanisms whereby sense and nonsense are learnt are the same” (p. 493).

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