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This paper has benefitted greatly from extensive comments on earlier drafts by Bela Balassa and Hugh Collier. Most of the numerical computations were made or checked by the Bank's Statistical Services Division.

Prepared by: Benjamin B. King
Senior Adviser
"Whom have you got on board?" said I. Said he, 'Astrologers, fortune-tellers, alchymists, rhymers, poets, painters, projectors, mathematicians, watch-makers, sing-songs, musicianers, and the devil and all of others that are subject to Queen Whims. They have very fair legible patents to show for it, as anybody may see'. Panurge had no sooner heard this, but he was upon the high-rope, and began to rail at them like mad."

"The Heroic Deeds of Gargantua and Pantagruel"

By François Rabelais
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**ANNEXES**

1. Proofs of Identities in Section II
2. Initial Values
3. Properties of the Accumulation Function
4. Notation, Brevity and Accuracy
5. Investment with a Two-year Lag
I. Introduction

The original purpose of this note was to describe the mechanics of a process which is anything but mechanical in real life, namely, the transfer of resources from one country to another. As the transfer brings about at the same time an increase in the indebtedness and the income of the recipient country, it is possible to bring into a model of the process three or four different elements of growth, each interacting on the other. The model, even though highly aggregated, can easily become quite complicated.

Whether complexity is a necessary evil or not depends on the nature and purpose of the model. It is unavoidable in models which attempt to integrate the behavior of a fairly large number of variables and which need sophisticated econometric techniques to create and refine them. The principal test of these models is not whether they readily illuminate some basic truth or other; it is whether they work, i.e. whether they can be used to forecast or to plan. In less developed countries, where data are scarce and unreliable, it remains to be seen what can be done, even in the short run.

At the other end of the spectrum is the type of model whose purpose has been suggested by Professor Mahalanobis: "I find simple models useful in planning (just as the thermo-dynamical approach is useful in physics) in revealing the broad characteristics of the system under consideration without getting lost in the details." 1/ Simple, expository models of

this type are useful, but dangerous tools. It is easy to lose sight of
the fact that, by expressing explicitly one set of relationships, alter-
native ones are implicitly denied. It may be even easier to do this, if
one succumbs to the temptation to complicate in the belief that this is
a concession to reality. One or two concessions of this kind do not
necessarily bring one any closer to reality and may at the same time
liberate one from knowing what one is doing. 1/ Preoccupation with the
sensitivity of one variable to variations in another may preclude an
investigation of the sensitivity of a given variable to changes in the
model itself.

The urge to complicate is often provoked by the temptation to use
expository models as predictive tools, in spite of the innumerable
assumptions inherent in the model. Besides smacking of pseudo-science,
this sort of procedure misses one of the main purposes of the expository
model, namely, as a policy instrument, a way of illuminating choices.
This was the object of the Mahalanobis model, exposing the choice
between producing capital goods to produce capital goods and producing
capital goods to produce consumer goods in a closed economy. It does
not and was not intended to do the same thing for an open economy. If
it was so used, implicitly or explicitly, it was not the fault of the
model.

1/ As the Interim Report of the Working Party on Assistance Require-
ments, DAC/BA(65)7, pointed out: "Consequently, while a fair degree of
disaggregation may help to make a model more realistic and reliable, it
is important to judge the point beyond which disaggregation begins to
give only a spurious impression of precision" (page 29).
To use an expository model as though it had oracular significance is to miss the fact that in almost all economies—and especially less developed ones—almost everything is subject to change to a greater or lesser extent, by direction or indirection, at the will of the government, if it has one, and sometimes whether it likes it or not.

Although the first purpose of this paper is to try to reduce the complications which appear inherent in an aid-growth model, it has, therefore, a secondary purpose. This is to examine in a modest way the influence of alternative hypotheses on any conclusions one might be tempted to draw. The examination is concentrated on the investment-savings gap to the exclusion of the export-import gap. If the latter is not mentioned, it is not because it can be ignored, but because there is some advantage in taking one thing at a time.

The model used, with its variants, is based on the simple Harrod-Domar prototype. However, there is an ambiguity in the savings function of particular importance in the present context. We may consider a savings rate fixed in relation to gross domestic product or to gross national product. Savings of 10% of gross domestic product are equivalent to savings of 7.3% of gross national product if the outflow of investment income at the time amounts to as much as 3% of gross national product. But the equivalence is only an equivalence at a point of time. Models based on one or the other savings relationship behave quite differently. It is this difference between the two versions with which we shall mainly be concerned.
It would be wrong to say that the difference has not been noticed, but the implications appear not to have been fully considered in the literature. Some models do not introduce the return flow of investment income at all. Others opt for one or other version without exploring the alternative. 1/

In either version external assistance has the same immediate or direct effect on income. But the incremental savings out of this incremental income, compounded over time, are different; thus the difference lies in this indirect effect on income.

We should not exaggerate the difference over a short period. But since the process is cumulative, it becomes more and more marked. Quite different conclusions could be drawn about the productivity of external assistance, the proper terms on which it should be given and the nature of the so-called debt cycle. If there is a conclusion, it is that, so long as we know little about both the nature of the savings function and the values to be put on the variables, modesty is the best policy.

The remainder of this paper falls into four parts. Section II introduces ad hoc a new notation for what has been termed the "accumulation function". While this section certainly imposes on the reader the

---

1/ Income arising from capital inflow is evidently taken as zero in H.B. Chenery and A.M. Strout, "Foreign Assistance and Economic Development", American Economic Review, LVI, Number 1, Part 1 (September 1966); otherwise the first equation in their article, which makes gross national product equal to gross domestic savings plus consumption, is not true. The domestic savings version is used in Dragoslav Avramovic and Associates, Economic Growth and External Debt, (Baltimore: Johns Hopkins Press 1964) on the basis of a very brief argument.
task of learning a new technique, it will, hopefully, save him time, if he has the courage to continue. Section III uses this technique to establish certain rules for building models of both types, given any pattern of external aid as a starting point or, alternatively, any growth pattern for gross product.

Section IV compares the two versions and explains in detail, with numerical examples, how the difference arises. It has some tentative things to say on their relative merits. In the final section V, the possibilities for elaboration and complication of the models are suggested, if, it should be said again, there is good reason to do so.
II. The Accumulation Function: A Necessary Tool

The purpose of this section is to introduce a practical working tool. Any model in which one or more variables grows at a constant rate per annum (or any other period) will necessarily involve certain mathematical expressions. The more interaction between variables, the more complicated and tedious to the reader these expressions will be and the easier it will be to lose sight of what is going on. If we can devise a simple notation and simple rules to deal with them, so much the better.

Two of these expressions are familiar to anyone who uses compound interest tables:

- Amount at compound interest \( (1 + i)^n \)
- Amount of an annuity \( \frac{(1 + i)^n - 1}{i} \)

The first formula expresses what an amount of 1 will grow to in \( n \) periods whether it be an investment growing at an interest rate of \( i \) per annum or an economy at the same rate of growth. \(^1\)

The second formula gives the amount that \( n \) successive investments of 1 each per annum will amount to at the time of the last investment if each grows at interest rate \( i \). Specifically:

\[
\sum_{t=1}^{n} (1 + i)^{t-1} = \frac{(1 + i)^n - 1}{i} \quad (1)
\]

\(^1\) Throughout this paper interest is assumed to be paid annually.
However, we might and, in fact, we will wish to consider a series of investments, which are not equal but are themselves growing at a different rate, say, h. Then the question will be what is the accumulated sum of these successive and increasing investments with their accumulated interest. We may represent the problem thus:

<table>
<thead>
<tr>
<th>Year</th>
<th>Investment</th>
<th>Compound interest factor in year n</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$(1+i)^{n-1}$</td>
<td>$(1+i)^{n-1}$</td>
</tr>
<tr>
<td>2</td>
<td>$(1+h)$</td>
<td>$(1+i)^{n-2}$</td>
<td>$(1+h)(1+i)^{n-2}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>n</td>
<td>$(1+h)^{n-1}$</td>
<td>1</td>
<td>$(1+h)^{n-1}$</td>
</tr>
</tbody>
</table>

The investment 1 in year 1 has $(n-1)$ years to grow; hence it is multiplied by $(1+i)^{n-1}$. The investment in year 2 is $(1+h)$, but it only has $(n-2)$ years to grow; hence the compound interest factor is $(1+i)^{n-2}$. And so on. The sum of the expressions in the last column is:

$$
\sum_{t=1}^{n} (1+h)^{t-1} (1+i)^{n-t} = \frac{(1+i)^n - (1+h)^n}{(1+h) - (1+i)}
$$

This is, in fact, a more generalized version of the amount of an annuity, since the latter can be derived simply by putting $h = 0$.

It is clear from the symmetry of the expression on the right hand side that we can substitute $i$ and $h$ for each other. A series of
investments growing at a rate \( i \) per annum, which accumulate interest at \( h \) per annum, will grow to the same amount. Specifically:

\[
\sum_{t=1}^{n} (1 + i)^{t-1}(1 + h)^{n-t} = \frac{(1 + h)^n - (1 + i)^n}{(h - i)} \tag{3}
\]

The right-hand side of either (2) or (3) may be written as follows:

\[
\frac{(1 + h)^n}{(h - i)} + \frac{(1 + i)^n}{(i - h)} \tag{4}
\]

Expressions like this and even more complicated ones will recur constantly. This is why we need a notation.

**Notation and Definition of the Accumulation Function**

We shall now define the accumulation function and the notation to be used as follows:

<table>
<thead>
<tr>
<th>Description</th>
<th>Notation</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>First generation</td>
<td>( I_n )</td>
<td>( (1 + i)^n ) \tag{5}</td>
</tr>
<tr>
<td>Second generation</td>
<td>( IH_n )</td>
<td>( \frac{(1 + i)^n}{(i - h)} + \frac{(1 + h)^n}{(h - i)} ) \tag{6}</td>
</tr>
</tbody>
</table>

The first of these is simply the compound interest formula. The second is the expression (4) we met just above. It follows from (3) that:

\[
\sum_{t=1}^{n} I_{t-1} (1 + h)^{n-t} = IH_n \tag{7}
\]

This is the rule for proceeding from the first generation to the second. We shall also apply the same rule in defining a third generation.
We proceed from the second to the third as follows:

\[ \sum_{t=1}^{n} I_{H_{n-1}} (1 + j)^{n-t} = I_{H_{n}}. \]

(8)

This third generation function may also be expressed in the same way as the first and second generation ones as follows (see Annex I for proof):

\[ \frac{I_{H_{n}}} {I_{H_{n}}} = \frac{(1 + i)^{n}} {(1-h)(1-j)} + \frac{(1 + h)^{n}} {(h-i)(h-j)} + \frac{(1 + j)^{n}} {(j-h)(j-i)}. \]

(9)

There is, in fact, a regular pattern, which would enable us to build up each generation from the previous one by accumulating the latter as in (7) or (8) and also to write out directly an expression for the consequent sum. But we shall only need three generations here. More generalized rules are given in Annex 3.

It was possible to explain, in reasonably everyday terms, the significance of the second generation function as the accumulated amount of a series of investments growing at a particular rate, together with interest compounded at a different rate. It is not possible to describe the third generation function in the same way. We can do no more than express in words what equation (8) says in symbols, namely, that the third generation function is the result of investing an amount equal to the second generation function at yet a third rate of return. It will be observed from the symmetry of equation (9) that the order in which the three rates of return appear makes no difference. \( I_{H_{n}} \) is equal to \( I_{H_{n}} \). Subsequently, following sound theatrical practice, the characters will usually be placed in the order of their appearance,
but this is not actually necessary.

It should be emphasized that we have merely introduced a notation for the purpose of brevity and, hopefully, clarity. \(^1\) The notation \(H_n\) does not mean, in any sense, that \(H\) is multiplied by \(I\) and \(J\). It only means what it is defined to mean, namely the sum expressed in (9). Rules can be developed for treating these underlined letters as a kind of mathematical operator, but it is not necessary to go into them here.

One advantage of the notation, is that, every time there is an additional accumulation process, all one has to do is to add another letter.\(^2\) With a little familiarity, working with the function in this form can become a fairly simple business. Care must be exercised as to whether the suffix should be \((n-1)\) or \(n\) or \((n+1)\), but this can usually be checked by putting \(n = 0\) or \(1\).

Some useful identities

A pair of identities, which will prove useful, is as follows:

\[
\frac{I_n - H_n}{(i-h)} = IH_n \tag{10}
\]

\[
\frac{H_n - H_n}{(i-h)} = IHJ_n \tag{11}
\]

The first of these is evident from equation (6). The second, which is not so obvious, is proved in Annex 1.

---

\(^1\) An application to published material is given in Annex 4.

\(^2\) There is a complete parallel with exponential functions and successive integration. \(H_t = e^{ht}\); \(HI_t = e^{ht} - e^{it}\); and so on.
Initial values

It is useful to know the initial values of these expressions which are as follows:

\[ I_1 = 1 + i \quad IH_2 = 2 + i + h \quad IHJ_3 = 3 + i + h + j \]
\[ I_0 = 1 \quad IH_0 = 1 \quad IHJ_0 = 1 \]
\[ IH_0 = 0 \quad IHJ_1 = 0 \]
\[ IHJ_0 = 0 \]

Many of these are self-evident; the remainder are explained in Annex 2.

Zero growth rate

We shall designate a zero growth rate by an 0 in the function, thus, putting \( i = 0 \) in (5), (6) and (9):

\[ 0_n = 1 \quad \tag{12} \]
\[ \frac{OH_n}{h} = \frac{1}{h} + \frac{(1 + h)^n}{h} \quad \tag{13} \]
\[ \frac{OHJ_n}{hj} = \frac{1}{hj} + \frac{(1 + h)^n}{h(h - j)} + \frac{(1 + j)^n}{j(j - h)} \quad \tag{14} \]

The second of these is simply the formula (1) for the amount of an annuity, since it equals:

\[ \frac{(1 + h)^n - 1}{h} \]
Equal growth rates

In the special case when \( h = i \), the formulae in (6) and (9) plainly do not work. Going back to (7), however, it is clear that:

\[
\sum_{t=1}^{n} (1 + h)^{t-1} (1 + h)^{n-t}
\]

\[
= \sum_{t=1}^{n} (1 + h)^{n-1}
\]

\[
= n (1 + h)^{n-1}
\]

(15)

It is fairly easy to show in a similar way that:

\[
\sum_{t=1}^{n} (1 + h)^{n-2}
\]

(16)

Numerical examples

This is all we need to proceed to the next step—consideration of the models. But in order to give the reader an idea of the order of magnitude of these unfamiliar expressions, values of the second generation function are shown in the accompanying Table 1 for a period of 20 years and for various rates of interest and growth.

As we explained earlier, the second generation expression is the sum of 20 (in this case) elements each growing at a kind of composite compound interest, part of the time at one rate and part at another. If we take 3 and 5 percent as an example, one might expect, therefore, that the value of the function at these rates would be approximately the same as at \( h \) and \( i \) percent, which indeed it is. As noted earlier, the latter is equal to 20 times the amount of \( l \) at compound interest.
for 19 years; and similarly for other values along the diagonal where both rates are the same.

It is impossible to perform the same service for the third generation expression because one needs three dimensions. It is, however, possible to indicate orders of magnitude by showing the values of the three generations along the diagonals, i.e. when all growth elements are the same:

<table>
<thead>
<tr>
<th>Formula</th>
<th>At 5% in 20 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>I = (1 + i)</td>
<td>2.7</td>
</tr>
<tr>
<td>II&lt;sub&gt;n&lt;/sub&gt; = n(1 + i)&lt;sup&gt;n-1&lt;/sup&gt;</td>
<td>50.5</td>
</tr>
<tr>
<td>III = ( \frac{n(n - 1)}{2} (1 + i)^{n-2} )</td>
<td>457.3</td>
</tr>
</tbody>
</table>
Table 1

Values of the second generation function at various rates (20 years)

<table>
<thead>
<tr>
<th>Growth rate (percent)</th>
<th>Interest rate (percent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20.0 22.0 24.3 26.9 29.8 33.1 36.8 41.0 45.8</td>
</tr>
<tr>
<td>1</td>
<td>22.0 24.2 26.6 29.3 32.4 35.6 39.7 44.2 49.2</td>
</tr>
<tr>
<td>2</td>
<td>24.3 26.6 29.1 32.0 35.3 38.9 43.0 47.7 52.3</td>
</tr>
<tr>
<td>3</td>
<td>26.9 29.3 32.0 35.1 38.5 42.4 46.7 51.6 57.1</td>
</tr>
<tr>
<td>4</td>
<td>29.8 32.4 35.3 38.5 42.1 46.2 50.8 56.0 61.7</td>
</tr>
<tr>
<td>5</td>
<td>33.1 35.8 38.9 42.4 46.2 50.5 55.4 60.8 66.9</td>
</tr>
<tr>
<td>6</td>
<td>36.8 39.7 43.0 46.7 50.8 55.4 60.5 66.3 72.7</td>
</tr>
<tr>
<td>7</td>
<td>41.0 44.2 47.7 51.6 56.0 60.8 66.3 72.3 79.1</td>
</tr>
<tr>
<td>8</td>
<td>45.8 49.2 52.3 57.1 61.7 66.9 72.7 79.1 86.3</td>
</tr>
</tbody>
</table>
III. Savings Models; Rules of the Game

The model used is, as we said, based on the simple Harrod-Domar prototype. We shall consider the version in which the savings rate is fixed in relation to gross domestic product and one in which it is fixed in relation to gross national product. We shall have to devise rules for both types, which will form a sort of do-it-yourself kit.

It is unfortunately necessary to introduce some new terminology. In the simplest type of model without external aid an economy will grow at a rate determined by the savings rate (s) and the capital-output ratio (k); s = kj. This rate of growth, "j", is an important element in the model. Since it is also not the actual rate of growth, there will be occasions when it has to have a name. It has been called, as sparingly as possible, the "intrinsic rate of growth". It can have an initial and a marginal version.

Two preliminary remarks must be made about this exercise. First, debt is used as a term of art. It includes conventional and non-conventional loans, grants and equity investment. Grants do not present a problem since they are debts at zero interest; "outstanding debt" then includes accumulated grants. Equity investment, however, is another matter. Unless we believe that the return on equity investment is reasonably constant over time, the model needs modification.

Secondly, everything in the model is in gross terms, i.e. not net of depreciation. Strictly speaking, one should then make an allowance

1/ Generally, we use s and j for the national savings version, s' and j' for the domestic.
for replacement of capital. This presents no problem when investment grows at a constant rate because replacement can easily be incorporated in the capital-output ratio. This is not true at variable growth rates. This is a lacuna, for which one can think of various weak excuses, but it is perhaps better to just plead guilty and pass on.

Some definitions and relationships

We shall define the inflow of external capital before deduction of investment income payments abroad as the flow of financial resources and the inflow of external capital after deduction of investment income payments abroad as the flow of real resources. We denote these as \( F \) and \( T \) respectively. \(^1\) If \( D \) is outstanding debt at the beginning of the year and \( i \) the interest rate, then:

\[
F = T + iD
\]

If \( V = GDP \) and \( Z = GNP \), then

\[
V = Z + iD
\]

If \( S' = \) domestic savings and \( S = \) national savings,

\[
S' = S + iD
\]

If \( I = \) investment,

\[
I = S' + T = S + F
\]

Thus, in the two models which follow, the following are associated:

(i) Domestic savings and the flow of real resources

(ii) National savings and the flow of financial resources

---

\(^1\) The inflow of external capital is defined net of amortization payments here and throughout the rest of the paper.
Domestic savings model (constant savings rate)

We can approach the construction of a model in either of two ways. We can assume a certain pattern of behavior of GDP, e.g. a constant growth rate, and see what pattern of external aid is required to achieve this. Alternatively, we can assume a certain pattern of external flow of resources and see what happens to GDP as a consequence. Purely for the purposes of building the model, we shall do the latter; once the rules have been established, the assumptions can readily be varied.

Specifically, we assume that the flow of real resources \( T \) grows at a constant rate \( r \) per annum:

\[
T_n = T_0 R_n
\]

However, since we shall have occasion to divide \( T_0 \) by the capital-output ratio \( k \), we shall introduce a constant \( a' \), such that \( T_0 = ka' \). Hence:

\[
T_n = ka' R_n
\]

The increase in indebtedness equals the transfer of financial resources \( F \), which equals the transfer of real resources \( T \) plus interest on outstanding debt \( iD \). Assuming there is no initial debt, debt outstanding therefore equals the total of these transfers of real resources accumulated at compound interest. If the interest rate is \( i \), this equals:

\[
D = \sum_{t=0}^{n-1} T_t (1 + i)^{n-t-1}
\]

\[
= \sum_{t=1}^{n} ka' R_{t-1} (1 + i)^{n-t}
\]

\[
= ka' RI_n
\]
This may be checked by putting \( n = 0 \) and \( I \) above. Since \( R_{i0} = 0 \) and \( R_{i1} = 1 \), debt at the beginning of year 0 is zero and at the beginning of year 1 is the initial inflow \( ka' \). This is correct.

Now, to determine gross domestic product, we may consider separately what would happen to the economy on its own and what contribution external aid makes if there is a constant domestic savings rate of \( s' \). Assuming a one-year lag between investment and income, \( \frac{1}{k} \) the increment in GDP (\( dV \)) without inflow of capital would be given by:

\[
k dV = s' V
\]

Putting \( s' = kj' \), where \( j' \) is the intrinsic rate of growth,

\[
dV = j' V
\]

In other words, if there were no inflow of real resources, the economy would grow at a rate \( j' \):

\[
V_n = V_0 j'_{t_{n}}
\]

In order to determine the addition to GDP as a result of external investment, we shall first consider the contribution of each annual investment separately. The external investment in year \( t \) equals \( ka'R_t \). The increment in income in year \( (t + 1) \) as a result of this investment equals \( a'R_t \). But, as a result of domestic savings out of this income, it will grow at a rate \( j' \) for the rest of the period, i.e. the remaining \( n - (t + 1) \) years. Hence the contribution

---

1/ This assumption is made throughout. However, see Annex 5 for a different assumption.
to income in year \( n \) of external investment in year \( t \) is:

\[ a' R_t (1 + j')^{n-t-1} \]

The sum of the contributions of all investments from year \( t = 0 \) to year \( t = n-1 \) will be:

\[
\sum_{t=0}^{n-1} a' R_t (1 + j')^{n-t-1}
\]

\[
\sum_{t=1}^{n} a' R_{t-1} (1 + j')^{n-t}
\]

\[ = a' \frac{R_j}{n} \]

Hence the complete expression for \( V_n \) will be:

\[ V_n = V_0\frac{J'}{n} + a' \frac{R_j}{n} \]

National savings model (constant savings rate)

There are two points of difference in this version. First, transfer of financial resources (\( F \)) is itself equal to the increment in debt. So it is not necessary to accumulate this transfer at interest. Again we shall make an assumption purely for the purpose of building up the model, namely,

\[ F_n = F_0 B_n \]

\[ = ka_{R_h} \]
The constant \( a \) is introduced, as before for convenience.

In this event, debt will accumulate at zero interest rate,

\[
D_n = kaR_O
\]

Secondly, capital imported must produce not only the required product but also the interest to pay for the capital. A capital amount \( k \) produces:

\[
\begin{align*}
\text{GDP} & \quad 1 \\
\text{Less interest} & \quad ki \\
\text{GNP} & \quad 1 - ki
\end{align*}
\]

The increment in GNP as a result of imported capital of \( k \) is, thus, not 1 but \( 1 - ki \). Hence, in terms of GNP, the effective capital-output ratio for imported capital is \( k/(1 - ki) \). Otherwise the analysis is the same as before. If there is a constant national savings rate of \( s \) and a corresponding intrinsic rate of growth of \( j \), debt and product are as follows:

\[
\begin{align*}
\text{Debt outstanding} & \quad D_n = kaR_O \\
\text{GNP} & \quad Z_n = Z_0J_n + a(l - ki)R_J
\end{align*}
\]
The basic rules

The two results are shown in Table 2. This establishes the basic rules for deducing from more complicated expressions for external aid the related quantities for debt and gross product. From this set of blocks, we can build up an expression on either savings basis for the same set of assumptions about the transfer of resources. We shall also drop the subscript \( n \). Henceforth \( \bar{R} \), for example, means \( R_n \).

We can, for example, make a comparison between the two models on a common assumption that the flow of financial resources \((F) = ka \bar{R}\). In that event, the flow of real resources \((T)\) equals \( F \) less interest, i.e.

\[
T = ka \bar{R} - kai \bar{R}O
\]

\[
= ka (\bar{R} - i \bar{R}O)
\]

Thus the formula for the transfer of real resources, instead of being a simple expression like \( ka \bar{R} \), is more complicated. But the rule for deducing the addition to GDP is just the same, namely, division by \( k \) and accumulation at rate \( j' \). Hence GDP will, in this event, be:

\[
V = V_0j' + a(Rj' - iRj')
\]

*/ Inspection of the formula shows that if \( i \) exceeds \( r \), \( T \) must become negative.*
## Table 2

### Constant Savings Rate

<table>
<thead>
<tr>
<th></th>
<th>National savings basis</th>
<th>Domestic savings basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt outstanding at beginning of year */ (Dₙ)</td>
<td>ka ROₙ</td>
<td>ka'RJₙ</td>
</tr>
<tr>
<td>Debt interest */ (iDₙ)</td>
<td>kai ROₙ</td>
<td>ka'iRJₙ</td>
</tr>
<tr>
<td>Flow of financial resources (Fₙ)</td>
<td>ka Rₙ</td>
<td></td>
</tr>
<tr>
<td>Flow of real resources (Tₙ)</td>
<td></td>
<td>ka'RJₙ</td>
</tr>
<tr>
<td>GNP (Zₙ)</td>
<td>Z₀Jₙ₊a₁(₁-ki)RJₙ</td>
<td></td>
</tr>
<tr>
<td>GDP (Vₙ)</td>
<td>V₀J'ₙ₊a'RJ'ₙ</td>
<td></td>
</tr>
</tbody>
</table>

*/ Assuming D₀ = 0.
By adding, or subtracting debt interest, where appropriate, to obtain GDP or GNP in the two columns, we can now construct Table 3, in which the flow of resources is the same on each savings basis, but, of course, the behavior of the models is different.

Table 3

<table>
<thead>
<tr>
<th>National savings basis</th>
<th>Domestic savings basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt *^/</td>
<td>ka RO</td>
</tr>
<tr>
<td>Debt interest *^/</td>
<td>kai RO</td>
</tr>
<tr>
<td>Flow of financial resources (F)</td>
<td>ka R</td>
</tr>
<tr>
<td>GNP (Z)</td>
<td>Z_{J+a(1-k_i)RJ} V_{OJ'+a(R'_{i-ROJ'})} - kai RO</td>
</tr>
<tr>
<td>Flow of real resources (T)</td>
<td>ka (R_{i-RO})</td>
</tr>
<tr>
<td>GDP (V)</td>
<td>Z_{OJ+a(1-k_i)RJ+kaiRO} V_{OJ'+a(R'_{i-ROJ'})}</td>
</tr>
</tbody>
</table>

\*\^/ Assuming D_0 = 0. In that event, of course, Z_0 = V_0.
Target rate of growth and initial savings rate

This simple model is all we shall need to make a comparison between the two versions in the next section. The model is capable, however, of further elaboration, which for the most part will be deferred until later. However, we should make good the promise that the model can, so to speak, be put into reverse. One can start with a target rate of growth of income and determine the necessary growth of aid. This will be done for the national savings version here.

If the target rate of growth is \( r \),

\[
Z = Z_0 R
\]

Hence, if \( F = kaR \), then

\[
Z_0 I + a(l - ki)R = Z_0 R
\]

It can readily be seen that this will be the case, if

\[
\frac{a(l - ki)}{r - j} = Z_0
\]

The required pattern of external aid is:

\[
F = \frac{Z_0 k(r-j)R}{1 - ki}
\]

This will be positive or negative according to whether \( r \) is greater or less than \( j \). Ordinarily, however, such target rate of growth models postulate an initial savings rate lower than the marginal one. The consequences can be put fairly simply. But we must first forget about external assistance and see what happens to an economy on its own.
If we assume an initial savings rate \( s_o \), savings and therefore investment are less than in the previous model in the initial year and in each subsequent year by an amount \( Z_o(s - s_o) \). If there were to be an external investment of this amount each year, the growth of GNP would be the same as before (i.e. at a rate of \( j \)). The effect is thus similar to that of external investment except that it is negative.

The consequences are the same as if there were a negative transfer of financial resources (without interest) equivalent to:

\[
F = -Z_o(s - s_o)
\]

If we define an initial intrinsic rate of growth by \( s_o = kj_o \), the adjustment to GNP is found by applying the rule of dividing by \( k \) and accumulating at a rate \( j \). Hence the adjustment will be:

\[
- Z_o(j - j_o)Oj
\]

In toto, therefore:

\[
Z = Z_oOj - Z_o(j - j_o)Oj
= Z_o + j_o Oj
\]

If we wish to make good this shortfall of savings, there must be external aid of an equivalent amount plus whatever is needed to take care of interest. Hence there must be an external investment of:

\[
F = \frac{Z_o(j - j_o) O}{1 - ki}
\]
We can now combine the two results. If there is an initial savings rate $s_0$ and the objective is a target rate of growth of $r$, the required external investment is:

$$F = \frac{Z_0(s - s_0)}{1 - ki} + \frac{Z_0k(r - j)R}{1 - ki}$$

The first part raises the growth rate to $j$ and the second raises or lowers it from $j$ to $r$. 
IV. A Comparison of the Models

We have said that the models on a national savings basis or a domestic savings basis behave differently. In this section we shall compare their behavior, try to explain why there are differences and suggest, rather tentatively, which of them is more useful.

Four numerical examples have been worked out, which are shown in Tables 6 to 9. They are all based on the simplest versions of the models, that is, without initial conditions. In order to illustrate what happens under different circumstances, the parameters are changed from one example to another. But, in each example, the behavior of the two models is compared under conditions which are as identical as possible.

In each example, all but one of the parameters are common to both. The initial flow of financial resources, the rate at which it grows and the interest rate are all the same. The same capital-output ratio is used for both versions and the same initial volume of savings. This means that initially the domestic savings ratio and the national savings ratio are the same, there being no interest on debt. From then on, if one of them is fixed, they part company.

In the first example (Table 4), the intrinsic rate of growth of the economy (j) is higher than either the rate of growth of external aid (r) or the interest rate (i). Ultimately external aid becomes negligible and so does interest on debt. If the national savings ratio is fixed, the domestic savings ratio becomes a little larger. If the
Table 4

Example 1

<table>
<thead>
<tr>
<th>Common features</th>
<th>Year 0</th>
<th>Year 10</th>
<th>Year 25</th>
<th>Year 50</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aid %/</td>
<td>3.0</td>
<td>4.4</td>
<td>8.0</td>
<td>21.3</td>
<td>⋯</td>
</tr>
<tr>
<td>Debt interest</td>
<td>⋯</td>
<td>1.1</td>
<td>3.7</td>
<td>13.7</td>
<td>⋯</td>
</tr>
</tbody>
</table>

National savings basis

<table>
<thead>
<tr>
<th></th>
<th>Year 0</th>
<th>Year 10</th>
<th>Year 25</th>
<th>Year 50</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>GNP: total</td>
<td>100.0</td>
<td>176.4</td>
<td>404.2</td>
<td>1543.6</td>
<td>⋯</td>
</tr>
<tr>
<td>external contr.</td>
<td>⋯</td>
<td>13.5</td>
<td>65.6</td>
<td>396.9</td>
<td>⋯</td>
</tr>
<tr>
<td>Savings ratio: national %</td>
<td>15.0</td>
<td>15.0</td>
<td>15.0</td>
<td>15.0</td>
<td>15.0</td>
</tr>
<tr>
<td>domestic %</td>
<td>15.0</td>
<td>15.0</td>
<td>15.8</td>
<td>15.7</td>
<td>15.0</td>
</tr>
<tr>
<td>Aid/GNP %</td>
<td>3.0</td>
<td>2.5</td>
<td>2.0</td>
<td>1.4</td>
<td>0</td>
</tr>
<tr>
<td>Debt int./GNP %</td>
<td>⋯</td>
<td>0.6</td>
<td>0.9</td>
<td>0.9</td>
<td>0</td>
</tr>
</tbody>
</table>

Domestic savings basis

<table>
<thead>
<tr>
<th></th>
<th>Year 0</th>
<th>Year 10</th>
<th>Year 25</th>
<th>Year 50</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>GNP: total</td>
<td>100.0</td>
<td>175.0</td>
<td>388.7</td>
<td>1399.1</td>
<td>⋯</td>
</tr>
<tr>
<td>external contr.</td>
<td>⋯</td>
<td>12.1</td>
<td>50.1</td>
<td>252.4</td>
<td>⋯</td>
</tr>
<tr>
<td>Savings ratio: national %</td>
<td>15.0</td>
<td>14.5</td>
<td>14.2</td>
<td>14.2</td>
<td>15.0</td>
</tr>
<tr>
<td>domestic %</td>
<td>15.0</td>
<td>15.0</td>
<td>15.0</td>
<td>15.0</td>
<td>15.0</td>
</tr>
<tr>
<td>Aid/GNP %</td>
<td>3.0</td>
<td>2.5</td>
<td>2.1</td>
<td>1.5</td>
<td>0</td>
</tr>
<tr>
<td>Debt int./GNP %</td>
<td>⋯</td>
<td>0.6</td>
<td>1.0</td>
<td>1.0</td>
<td>0</td>
</tr>
</tbody>
</table>

*/ Flow of financial resources.

\[
k = 3 \\
a \text{or} \ a' = 1 \\
s \text{or} \ s' = .15 \\
j \text{or} \ j' = .05 \\
i = .03 \\
r = .04 \]
domestic savings ratio is fixed, the national savings ratio becomes a little smaller. The only thing perhaps worth noting is that, for a given fixed national savings ratio, the external contribution to GNP is substantially higher than for the same fixed domestic savings ratio. That there would be a difference is, of course, to be expected.

In the second example (Table 5), the intrinsic rate of growth of the economy \((j)\) is now less than the rate of growth of aid \((r)\), but still higher than the interest rate \((i)\). In this event, aid gradually increases as a proportion of GNP but there is a limit. Furthermore, as a consequence, debt interest does the same. Thus the gap between the two savings ratios widens continuously. On the domestic savings basis, the gap widens considerably more, because aid as a percentage of GNP grows faster and so therefore does interest on debt.

It is worth noting that the external contribution to GNP is not very different for either model from what it was in the previous example. This follows from two facts about the parameters chosen. First the initial amount of aid, when divided by the capital output ratio, i.e. \(a\), is the same. Secondly, the value of the external contribution depends heavily, though not wholly, on \(R\). But in fact \(r\) and \(j\) have simply been interchanged so that the value remains the same.

The third example (Table 6) is no different from the second, except for the interest rate, which has been raised from 3 to 5 percent and is now equal to the rate of growth of aid \((r)\). This means that the flow of real resources remains constant. On a national savings basis, this makes a perceptible difference to the external contribution to GNP, but
Table 5

Example 2

<table>
<thead>
<tr>
<th></th>
<th>Year 0</th>
<th>Year 10</th>
<th>Year 25</th>
<th>Year 50</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Common features</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Aid */</td>
<td>4.0</td>
<td>6.5</td>
<td>13.5</td>
<td>45.9</td>
<td>...</td>
</tr>
<tr>
<td>Debt interest</td>
<td>------</td>
<td>1.5</td>
<td>5.7</td>
<td>25.1</td>
<td>...</td>
</tr>
<tr>
<td><strong>National savings basis</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GNP: total</td>
<td>100.0</td>
<td>161.1</td>
<td>330.0</td>
<td>1094.4</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>------</td>
<td>13.1</td>
<td>63.4</td>
<td>333.7</td>
<td>...</td>
</tr>
<tr>
<td>Savings ratio: national %</td>
<td>16.0</td>
<td>16.0</td>
<td>16.0</td>
<td>16.0</td>
<td>16.0</td>
</tr>
<tr>
<td></td>
<td>16.0</td>
<td>16.8</td>
<td>17.4</td>
<td>17.9</td>
<td>18.2</td>
</tr>
<tr>
<td>Aid/GNP %</td>
<td>4.0</td>
<td>4.0</td>
<td>4.1</td>
<td>4.2</td>
<td>4.5</td>
</tr>
<tr>
<td>Debt Int./GNP %</td>
<td>------</td>
<td>0.9</td>
<td>1.7</td>
<td>2.3</td>
<td>2.7</td>
</tr>
<tr>
<td><strong>Domestic savings basis</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GNP: total</td>
<td>100.0</td>
<td>159.7</td>
<td>314.7</td>
<td>951.6</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>------</td>
<td>11.6</td>
<td>48.1</td>
<td>240.9</td>
<td>...</td>
</tr>
<tr>
<td>Savings ratio: national %</td>
<td>16.0</td>
<td>15.2</td>
<td>14.5</td>
<td>13.8</td>
<td>10.6</td>
</tr>
<tr>
<td></td>
<td>16.0</td>
<td>16.0</td>
<td>16.0</td>
<td>16.0</td>
<td>16.0</td>
</tr>
<tr>
<td>Aid/GNP %</td>
<td>4.0</td>
<td>4.1</td>
<td>4.3</td>
<td>4.8</td>
<td>10.6</td>
</tr>
<tr>
<td>Debt Int./GNP %</td>
<td>------</td>
<td>0.9</td>
<td>1.8</td>
<td>2.6</td>
<td>6.4</td>
</tr>
</tbody>
</table>

* Flow of financial resources

\[
k = 4
\]
\[
a \text{ or } a' = 1
\]
\[
s \text{ or } s' = .16
\]
\[
j \text{ or } j' = .04
\]
\[
i = .03
\]
\[
r = .05
\]
### Table 6

**Example 3**

<table>
<thead>
<tr>
<th>Common features</th>
<th>Year 0</th>
<th>Year 10</th>
<th>Year 25</th>
<th>Year 50</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aid */</td>
<td>4.0</td>
<td>6.5</td>
<td>13.5</td>
<td>45.9</td>
<td>....</td>
</tr>
<tr>
<td>Debt interest</td>
<td>----</td>
<td>2.5</td>
<td>9.5</td>
<td>41.9</td>
<td>....</td>
</tr>
</tbody>
</table>

#### National savings basis

<table>
<thead>
<tr>
<th>GNP: total</th>
<th>Year 0</th>
<th>Year 10</th>
<th>Year 25</th>
<th>Year 50</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>external contr.</td>
<td>100.0</td>
<td>159.9</td>
<td>324.2</td>
<td>1059.6</td>
<td>....</td>
</tr>
<tr>
<td>Savings ratio: national</td>
<td>16.0</td>
<td>16.0</td>
<td>16.0</td>
<td>16.0</td>
<td>16.0</td>
</tr>
<tr>
<td>domestic</td>
<td>16.0</td>
<td>17.3</td>
<td>18.4</td>
<td>19.2</td>
<td>20.0</td>
</tr>
<tr>
<td>Aid/GNP %</td>
<td>4.0</td>
<td>4.1</td>
<td>4.2</td>
<td>4.3</td>
<td>5.0</td>
</tr>
<tr>
<td>Debt int./GNP %</td>
<td>----</td>
<td>1.6</td>
<td>2.9</td>
<td>4.0</td>
<td>5.0</td>
</tr>
</tbody>
</table>

#### Domestic savings basis

<table>
<thead>
<tr>
<th>GNP: total</th>
<th>Year 0</th>
<th>Year 10</th>
<th>Year 25</th>
<th>Year 50</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>external contr.</td>
<td>100.0</td>
<td>157.5</td>
<td>298.7</td>
<td>821.5</td>
<td>....</td>
</tr>
</tbody>
</table>
| Savings ratio: national | 16.0 | 14.7 | 13.3 | 11.7 | ***/
| domestic | 16.0 | 16.0 | 16.0 | 16.0 | ***/
| Aid/GNP % | 4.0 | 4.1 | 4.5 | 5.6 | ***/
| Debt Int./GNP % | ---- | 1.6 | 3.2 | 5.1 | ***/

*/ Flow of financial resources.

***/ GNP becomes negative.

\[
\begin{align*}
  k &= 4 \\
  a \text{ or } a' &= 1 \\
  s \text{ or } s' &= 0.16 \\
  j \text{ or } j' &= 0.04 \\
  i &= 0.05 \\
  r &= 0.05 
\end{align*}
\]
most of it is explainable on the grounds that there is more interest to subtract. The various ratios in the table are somewhat higher, but not much.

However, on a domestic savings basis, the effect is drastic. In 50 years the external contribution to GNP is barely more than half what it was in the previous example. If one were to carry this model to its ultimate, but absurd, conclusion, debt interest would eventually become larger than GDP and GNP would become negative.

The fourth example (Table 7) is a somewhat different variant of the third. In this, there is no increase in aid from year to year ($r = 0$). The interest rate ($i$) is again 5 percent and the intrinsic rate of growth ($j$) 3 percent. On a national savings basis, nothing very dramatic happens. The external contribution to GNP, being based on a static volume of aid, is of course quite small in comparison to previous examples and eventually would become negligible.

On a domestic savings basis, somewhat the same thing happens as on a national savings basis, in that external aid ultimately becomes unimportant. But not only is the external contribution to GNP very small. By the end of 50 years it is actually negative and in fact will continue to be thereafter. It remains to be seen just how these rather strange results of the model occur.
Table 7

Example 4

<table>
<thead>
<tr>
<th>Common features</th>
<th>Year 0</th>
<th>Year 10</th>
<th>Year 25</th>
<th>Year 50</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aid %/</td>
<td>4.0</td>
<td>4.0</td>
<td>4.0</td>
<td>4.0</td>
<td>....</td>
</tr>
<tr>
<td>Debt interest</td>
<td>------</td>
<td>2.0</td>
<td>5.0</td>
<td>10.0</td>
<td>....</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>National savings basis</th>
<th>Year 0</th>
<th>Year 10</th>
<th>Year 25</th>
<th>Year 50</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>GNP: total</td>
<td>100.0</td>
<td>113.6</td>
<td>238.5</td>
<td>528.6</td>
<td>....</td>
</tr>
<tr>
<td>external contr.</td>
<td>------</td>
<td>9.2</td>
<td>29.2</td>
<td>90.2</td>
<td>....</td>
</tr>
<tr>
<td>Savings ratio: national %</td>
<td>12.0</td>
<td>12.0</td>
<td>12.0</td>
<td>12.0</td>
<td>12.0</td>
</tr>
<tr>
<td>domestic %</td>
<td>12.0</td>
<td>13.2</td>
<td>13.8</td>
<td>13.6</td>
<td>12.0</td>
</tr>
<tr>
<td>Aid/GNP %</td>
<td>4.0</td>
<td>2.8</td>
<td>1.7</td>
<td>0.8</td>
<td>0</td>
</tr>
<tr>
<td>Debit Int./GNP %</td>
<td>------</td>
<td>1.4</td>
<td>2.1</td>
<td>1.9</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Domestic savings basis</th>
<th>Year 0</th>
<th>Year 10</th>
<th>Year 25</th>
<th>Year 50</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>GNP: total</td>
<td>100.0</td>
<td>141.4</td>
<td>221.7</td>
<td>436.5</td>
<td>....</td>
</tr>
<tr>
<td>external contr.</td>
<td>------</td>
<td>7.0</td>
<td>12.4</td>
<td>-1.9</td>
<td>....</td>
</tr>
<tr>
<td>Savings ratio: national %</td>
<td>12.0</td>
<td>10.8</td>
<td>10.0</td>
<td>10.0</td>
<td>12.0</td>
</tr>
<tr>
<td>domestic %</td>
<td>12.0</td>
<td>12.0</td>
<td>12.0</td>
<td>12.0</td>
<td>12.0</td>
</tr>
<tr>
<td>Aid/GNP %</td>
<td>4.0</td>
<td>2.8</td>
<td>1.8</td>
<td>0.9</td>
<td>0</td>
</tr>
<tr>
<td>Debit Int./GNP %</td>
<td>------</td>
<td>1.4</td>
<td>2.3</td>
<td>2.3</td>
<td>0</td>
</tr>
</tbody>
</table>

# Flow of financial resources.

\[
k = 4 \\
a = 1 \\
v = .12 \\
j = .03 \\
i = .05 \\
r = 0 \]
How the difference arises

To understand what happens, we shall proceed by simple steps. First we take a single external investment and examine its consequences under the two different versions of the model.

In Tables 8 and 9, the contribution of aid to GNP is shown on the two savings bases in step-by-step fashion; Table 8 gives formulae and Table 9 a numerical example. The values of the parameters in the latter are the same as in Example 3 (Table 6). Under the first heading (A), the initial contribution to GNP of a single investment is shown; this is, of course, the same in both columns. Unless the capital-output ratio and the interest rate are both very high (e.g. 10 and .10), the contribution will be positive.

The contribution to national savings is, however, different. On a domestic savings basis, savings are taken as a proportion of GDP before deduction of investment income. But the addition to savings, so defined, is not fully available for investment; the outflow of investment income must first be subtracted. The additional amount available for investment is $k'(j' - i)$, which will be negative, if the rate of interest exceeds the intrinsic rate of growth. This happens because of the nature of the model. On a national savings basis, however, investment income is first subtracted from GNP and savings are taken as a proportion of the difference.

In the numerical example given in Table 9, investment income is larger than domestic savings (in the second column) and the net amount available for investment is actually negative. In this event, the
Table 8

The effect of a single investment

<table>
<thead>
<tr>
<th>National savings basis</th>
<th>Domestic savings basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>External aid (year 0)</td>
<td>ka</td>
</tr>
</tbody>
</table>

A. Consequences in year 1

<table>
<thead>
<tr>
<th>Addition to GDP</th>
<th>a</th>
<th>a'</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest</td>
<td>kai</td>
<td>ka'i</td>
</tr>
<tr>
<td></td>
<td>a(l - ki)</td>
<td>a'(l - ki)</td>
</tr>
</tbody>
</table>

Addition to domestic savings

<table>
<thead>
<tr>
<th>Interest</th>
<th>as + kei(l - s)</th>
<th>as'</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition to national savings</td>
<td>kai</td>
<td>ka'i</td>
</tr>
<tr>
<td>a(l - ki)</td>
<td>a's' - ka'i</td>
<td></td>
</tr>
<tr>
<td>= ka(j - is)</td>
<td>= ka'(j' - i)</td>
<td></td>
</tr>
</tbody>
</table>

B. Consequences in year (n + 1)

Addition to GNP

<table>
<thead>
<tr>
<th>a(l - ki)j</th>
<th>a'(l-ki)j' - a'i(l-s')Qj'</th>
</tr>
</thead>
<tbody>
<tr>
<td>of which:</td>
<td>a'(l - ki)j' - a'i(l-s')Qj'</td>
</tr>
<tr>
<td>Direct addition</td>
<td>a(l - ki)</td>
</tr>
<tr>
<td>Indirect addition</td>
<td>aj(l - ki) Qj</td>
</tr>
</tbody>
</table>
Table 9
The effect of a single investment

<table>
<thead>
<tr>
<th></th>
<th>National savings</th>
<th>Domestic savings</th>
</tr>
</thead>
<tbody>
<tr>
<td>External aid (year 0)</td>
<td>4.00</td>
<td>4.00</td>
</tr>
</tbody>
</table>

A. Consequences in year 1

<table>
<thead>
<tr>
<th></th>
<th>Basis</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition to GDP</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>Interest</td>
<td>0.20</td>
<td>0.20</td>
</tr>
<tr>
<td>Addition to GNP</td>
<td>0.80</td>
<td>0.80</td>
</tr>
<tr>
<td>Addition to domestic savings</td>
<td>0.328</td>
<td>0.16</td>
</tr>
<tr>
<td>Interest</td>
<td>0.20</td>
<td>0.20</td>
</tr>
<tr>
<td>Addition to national savings</td>
<td>0.128</td>
<td>-0.04</td>
</tr>
</tbody>
</table>

B. Consequences in year 25

<table>
<thead>
<tr>
<th></th>
<th>Basis</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition to GNP</td>
<td>2.05</td>
<td>0.41</td>
</tr>
<tr>
<td>of which:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Direct addition</td>
<td>0.80</td>
<td>0.80</td>
</tr>
<tr>
<td>Indirect addition</td>
<td>1.25</td>
<td>-0.39</td>
</tr>
</tbody>
</table>

Values

\[
\begin{align*}
    a \text{ or } a' &= 1 \\
    k &= \frac{1}{4} \\
    s \text{ or } s' &= 0.16 \\
    j \text{ or } j' &= 0.04 \\
    i &= 0.05
\end{align*}
\]
initial contribution of aid to income is positive, but as time passes, the cumulative effect of the negative addition to national savings gets larger and larger. Ultimately, the total contribution will become negative.

This can be seen under heading B, in which the effect of a single investment on GNP after n+1 years is shown. There is a direct addition (the original amount in the first year) and an indirect addition (the result of investments from the savings accumulated over the years).

Whether the indirect addition is positive or negative, it grows in absolute amount. In the first column the indirect addition is normally positive; in the second it is certainly smaller and will be negative if i is greater than j. In the numerical example in Table 9, where this is the case, the difference in the contribution of the single investment to GNP after 25 years is striking. The addition to national savings (in the first column) of 0.128 in a single year by itself gives rise—at a capital-output ratio of h—to additional income of 0.032. This is not large, but it happens every year and is cumulative at the growth rate j, here 4 percent. The amount of an annuity at this rate after 24 years is 39.1. Hence the indirect effect is by that time 1.25, much greater than the direct.

Similarly the very small negative effect on savings in the second column assumes considerable importance after 25 years. In 38 years the negative indirect effect would exceed the positive direct one.

In those cases, where i exceeds j, therefore, the model on a domestic savings basis tacitly assumes that the contribution to GNP of each external investment is ultimately negative. At any one time,
however, there will be some positive contributions from the most recent external investments and some negative ones from earlier ones.

Whether the overall external contribution to GNP becomes negative and whether this negative amount eventually is large enough to swamp the internal contribution depends on a variety of circumstances. Given the prior condition that the interest rate (i) is larger than the intrinsic rate of growth (j'), three broad cases may be distinguished:

(i) Slow growth of aid (roughly, r less than j');
(ii) Medium growth of aid (roughly, r between j' and i);
(iii) Fast growth of aid (roughly, r greater than i).

In each case we have said roughly, because these cases will have to be defined more precisely. The present distinction will do for a start.

If aid grows slowly, i.e. if r is less than j', the external contribution will ultimately become negative. ¹ But it will not necessarily be sufficiently large to swamp the internal contribution. For that to happen, the initial value of the flow of financial resources must be of a certain size. ² This makes sense. If aid is only growing slowly, the new positive contributions are not big enough to offset the accumulation of old negative ones. But unless aid reaches a certain size, this is unsatisfactory, but not disastrous, because the intrinsic growth of the economy is sufficient to offset the negative effects of aid.

In the second case, where r lies between j' and i, the external contribution is bound to become negative eventually and it is also

¹ See Table 3 (page 23). The terms in j' will become dominant and they are multiplied by the factor (1 - r/j').
² Specifically \( \frac{a}{V_0} > \frac{(j' - r) j'}{(i - j')} \).
bound to swamp the internal contribution.

The third case is where \( r \) exceeds \( i \). In fact, up to a point it is the same as the second. We really have to specify that \( r \) exceeds \( i \) by a certain amount. \( 1/ \) Up to this point, the external contribution is still negative and large enough to swamp the internal contribution ultimately. Beyond it, the external contribution is positive. This also makes sense. If aid is growing fast enough, the recent positive contributions outweigh the old negative ones. But it is rather like filling a bathtub with the stopper out.

The target rate of growth version

The model on a domestic savings basis has been used to describe what happens when a target rate of growth is postulated for GDP by Avramovic and Associates. \( */ \) In this model, there is no initial debt, but an initial savings rate. In our terminology the relevant magnitudes may be set forth as in Table 10.

In this description, attention is concentrated especially on the case where \( r \), the target rate of growth, is less than \( j' \), the

\[
1/ \text{Specifically } r - i > ki(r - j'). \text{ This again can be seen from an examination of the expression for GNP in Table 3.}
\]

Table 10

**Target rate of growth - domestic savings basis**

<table>
<thead>
<tr>
<th>Category</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt</td>
<td>( V_{0k} (j' - j'_o) OI + (r - j') RI )</td>
</tr>
<tr>
<td>Debt interest</td>
<td>( V_{0k} (j' - j'_o) OI + (r - j') RI )</td>
</tr>
<tr>
<td>Flow of financial resources</td>
<td>( V_{0k} (j' - j'_o) I + (r - j') (R + i RI) )</td>
</tr>
<tr>
<td>Flow of real resources</td>
<td>( V_{0k} (j' - j'_o) + (r - j') R )</td>
</tr>
<tr>
<td>GDP</td>
<td>( V_{0k} R )</td>
</tr>
</tbody>
</table>
intrinsic rate of growth. In this event, the flow of real resources ultimately becomes negative. What then happens to the economy at different rates of interest?

In this study, a "critical" rate of interest is identified, above which debt increases faster than GDP. The conditions can readily be identified from Table 10, namely, that $i$ is greater than $r$ and that the coefficient of $I$ in the formula for debt is greater than zero, i.e.

$$\frac{j' - j_0}{i} > \frac{j' - r}{i - r}$$

Or

$$i > \frac{r (j' - j_0)}{r - j'_{10}}$$

These are also the necessary and sufficient conditions for the flow of financial resources to increase at a rate faster than GDP. In this event, of course, GNP will become negative. This corresponds substantially to the slow growth example above, in which $i > j' > r$. To put it another way, debt becomes insupportable, given $i$ and $j'$, when $r$ exceeds a certain value. Below that, it is supportable.
There is also a corresponding case to the fast growth of aid or high rate of growth, i.e. the bathtub case, though this is not explicitly examined. In fact, if $r$ exceeds $i$, interest does not outstrip GNP as long as:

$$(r - i) > ki(r - j')$$

But this version of the model, like the other, rests on a foundation of implicit assumptions, which are not perhaps realized. One is the fact, already described, that whenever $i$ exceeds $j'$, the ultimate effect of each increment of aid is negative.

What, in fact, the model shows is that, if one assumes that any single increment of aid at an interest rate $(i)$ above the intrinsic rate of growth $(j')$ is ultimately self-defeating, then the cumulative effect is disastrous, if $i$ exceeds a certain critical rate. But is it legitimate to put this tacit assumption into the model? An expression of agnosticism

In plain everyday terms, what the model on a domestic savings basis is saying is this. If a borrower invests what he borrows and then consumes the resulting increment in income beyond the point where there is enough to pay for the interest, he is going to be in trouble. The size of his other income, the pace and the interest rate at which he borrows and the extent of his prodigality all determine the nature of the trouble, which may range from foolishness to disaster.

Whether people, governments or economies actually behave like that is open to serious question. Certainly, in the common case of a large enclave industry, whether it be oil or copper or bananas, it is only
our curious habits of national accounting which prevent us from realizing that a sizeable part of GDP or of export proceeds never sets foot on shore and is simply not available for consumption. What the country receives is not something less something (e.g. exports less investment income), but the difference (taxes and local expenditure). To build a model on the supposition that it is the former is to flout the facts.

This is not to say that there may not be other cases, where economies do exhibit the prodigal behavior described above. But evidence of such behavior is not evidence that it need or will continue indefinitely. There is no reason, ab initio, to abandon traditional accounting procedures, in which the cost of capital is treated as a cost by the recipient, either for behavioral or normative reasons. The justification for using a domestic savings basis has been put in these terms. 1/

"This form of presentation is required in order to explore the difference made by differences of interest rate, all the other variables remaining unchanged. If the marginal savings rate linked national (rather than domestic) savings to the increase of GDP, then a constant marginal national savings rate would in fact imply different savings efforts with different assumptions as to the rate of interest."

But this is a semantic trap. What is a saving effort? Whatever it is, why should one not expect it to vary with the rate of interest and, indeed, other circumstances as well? Where external assistance on a very large scale takes place, the effect may be to

reduce either national or domestic savings, in the conventional sense, to near zero or below it. 1/ Perhaps, under some circumstances, one should expect investment to be related in some way to total available resources rather than to be, as we formally reckon it, the sum of a "savings effort" and the inflow of capital from abroad.

A model on these lines, according to which every increment of external aid would immediately mean lower absolute savings, would be easy to construct. Another critical interest rate could be identified above which the external contribution of aid would become negative. Such a model would show interesting characteristics, but it would have no more validity than its own internal consistency.

We return again to the beginning. An expository model should expose whatever it has to expose, including its own, often severe, limitations. Before we can have confidence in using such models, even for illustrative purposes and over a comparatively short time, we need to know much more about the inputs. Otherwise, inserting rather arbitrary figures and grinding out an answer is just a game people play.

The next section is devoted to further complications of the model, but whether these complications are worthwhile is another matter. In practice the choice of savings functions, of capital-output and savings ratios at the margin and hence of the marginal intrinsic

1/ For example in Jordan, South Korea and, less recently, Israel.
rate of growth is likely to be rather wide. It is entirely possible that the range of results over a time-horizon such as 20 years will be extremely wide, making hazardous any conclusions about the amount of aid required to reach a specific income goal and the terms on which it is given. Over a somewhat shorter period—say 10 years—these marginal rates tend to be of less importance than the initial intrinsic rate of growth \((j_0)\). If there are ways and means of improving the flow of savings into more efficient capital investment and of increasing the efficiency of existing assets, these deserve as much attention. A model may help in this case to put the various alternatives in perspective.

The point is illustrated in Table 11 where, on a national savings basis, various ways of increasing the GNP target by an additional 2½ percent over and above the originally assumed increase of about 53 percent are shown. \(^1\) Since the contribution of aid is small rather drastic changes in the amount or the terms of aid are needed; on the other hand, fairly small changes in current efficiency or "performance" can have the same effect. This is all rather obvious, but needs to be emphasized, if the terms of aid are not invariably to be cast as the residual villain of the piece.

\(^1\) It is worth noting that the difference between the contribution of aid on a national and a domestic basis is about 25 (out of 1000) in this example, i.e. about the same order of magnitude as the other variations shown.
| Table 11 | 
|---|---|
| **GNP: Growth in 10 Years (National Savings Basis)** |  |
| **Initial value** | 1,000 |
| **Internal contribution to growth** | 395 |
| **External contribution** | 137 |
| **Total** | 1,533 |
| **Various ways of increasing GNP** | **Difference** |
| Increasing j from 6% to 7% | 25 |
| Increasing j₀ by 6% (i.e. to .0318) | 24 |
| Increasing efficiency of existing capital assets by 2% | 25 |
| Increasing aid by 20% (i.e. a = 12) | 27 |
| Reducing interest rate from 5% to zero | 25 |

---

| **Values** | **r** | .05 |
| **j** | .06 |
| **j₀** | .03 |
| **i** | .05 |
| **a** | 10 |
| **k** | 3 |
| **s** | .18 |
V. Further Complications

Having established the rules for the simple version of the models, it is mainly a mechanical matter to introduce additional complications. There are four main possibilities:

(i) Initial debt and interest on debt;
(ii) Initial savings rate, different from the marginal one, but applicable to a growing part of gross product;
(iii) A more complicated function for the growth of aid;
(iv) Differential capital-output ratios and interest rates.

The results of introducing the first two of these are shown in Table 12 for a national and a domestic savings basis respectively. Only the national savings model will be explained here; the reader can, if he so cares, work out how the formulae on a domestic savings basis are arrived at.

The first type of complication is not in fact a complication at all on a national savings basis. The only requirement is to add initial debt ($D_0$) and interest ($i_0 D_0$) to the previous values as in Table 12.

We have already explained in Section III that if there is an initial savings rate $s_0$ applicable to the initial value $Z_0$ of GNP, the necessary adjustment to GNP will be:

$$- Z_0 (j - j_0) Q_j$$
### Table 12

#### GNP on a National Savings Basis With Initial Conditions

<table>
<thead>
<tr>
<th>Component</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt</td>
<td>$D_0 + ka, RO$</td>
</tr>
<tr>
<td>Interest</td>
<td>$i_0D_0 + kia, RO$</td>
</tr>
<tr>
<td>Flow of financial resources</td>
<td>$ka, R$</td>
</tr>
<tr>
<td>GNP</td>
<td>$Z_0j - Z_0(j-j_0)Pj + a,(1-ki), RJ$</td>
</tr>
</tbody>
</table>

#### GDP on a Domestic Savings Basis With Initial Conditions

<table>
<thead>
<tr>
<th>Component</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt</td>
<td>$D_0 + i_0D_0\pi + ka', R\pi_n$</td>
</tr>
<tr>
<td>Interest</td>
<td>$i_0D_0\pi + ka', R\pi_n$</td>
</tr>
<tr>
<td>Flow of real resources</td>
<td>$ka', R$</td>
</tr>
<tr>
<td>GDP</td>
<td>$V_0j' - V_0(j' - j'_0)Pj' + a'Rj'$</td>
</tr>
</tbody>
</table>
However, there is no particular reason for us to stop at this point. The savings rate $s_o$ might be conceived of as applying to a growing amount of GNP, not a static one. For example, it might increase at the same rate as population, say "$P$". In this case, the annual reduction in savings would not be a constant $Z_0(s-s_o)Q$ but would be $Z_0(s-s_o)P$. The adjustment in GNP would be (see Table 12):

$$- Z_0(j-j_o)Pj$$

The third complication would amount to supposing that the flow of financial resources is not a simple function, but in the following form:

$$F = k(a_1 R_1 + a_2 R_2 + \ldots \ldots \text{etc. etc.})$$

As we have already pointed out, all we have to do in such a case is to add the comparable terms. Such modifications might be made, where public and private investment are expected to grow at different rates. In fact external public and private investment might also be expected both to have different capital-output ratios from that applied to the intrinsic growth of the economy and to carry different interest rates (or the equivalent). If these capital-output-ratios were $q_1$, $q_2$, etc. and the interest rates $i_1$, $i_2$, etc., then:

$$F = q_1 a_1 R_1 + q_2 a_2 R_2$$

The corresponding external contribution to GNP would be:

$$\frac{a_1 R_1 J}{1-q_1 i_1} \times \frac{a_2 R_2 J}{1-q_2 i_2}$$
But \( j \) would still be defined by \( s = kj \).

A further complication would be to suppose that the lag between investment and income is more than one year. A partial analysis of the case where the lag is two years will be found in Annex 5. It is partial, because it assumes a zero interest rate. To introduce interest would be to complicate matters further without really adding much to our knowledge. There are simpler rough devices which can be used, such as taking as one's unit of time not one year, but two or possibly more. In any long-term analysis, a single year has no particular merit as a unit. Capital-output ratios have intrinsic weaknesses, to which our lack of knowledge contributes. Excessive refinement of models is not a virtue.
ANNEX 1

Proofs of Identities in Section II

Equation (11)

From equation (10), we know that:

\[ \frac{I H_{t-1}}{b-1} = \frac{I_{t-1}}{1 - h} - \frac{H_{t-1}}{1 - h} \]

Therefore:

\[ \frac{I H_n}{J_n} = \sum_{t=1}^{n} \frac{I_{t-1}(1+j)^{n-t} - H_{t-1}(1+j)^{n-t}}{(i - h)} \]

\[ = \frac{I J_n - H J_n}{(i - h)} \]

This proves the identity in (11).

Equation (9)

Furthermore, by expanding as in (6), it follows that:

\[ \frac{I H_n}{(i-h)} - \frac{H J_n}{(i-h)} = \frac{I_n}{(i-j)(i-h)} + \frac{J_n}{(j-i)(i-h)} \]

\[ - \frac{H_n}{(i-h)(i-j)} - \frac{J_n}{(i-h)(j-h)} \]
The two coefficients of $J_n$ can be combined as follows:

\[
\frac{1}{(j-i)(i-h)} - \frac{1}{(j-h)(i-h)} = \frac{(j-h) - (j-i)}{(j-i)(j-h)(i-h)}
\]

\[
= \frac{1}{(j-i)(j-h)}
\]

Hence the right hand side of the previous expression is equal to:

\[
\frac{I_n}{(i-h)(i-j)} + \frac{H_{ij}}{(h-i)(h-j)} + \frac{J_n}{(j-i)(j-h)}
\]

This proves the identity in (9).
ANNEX 2

Initial Values

The values of $I_0$ and $I_1$ are self-evident. Those for $IH$ can be derived from the identity in equation (10) on page 10:

$$IH_0 = \frac{1 - 1}{i - h} = 0$$

$$IH_1 = \frac{(1 + i) - (1 + h)}{(i - h)} = 1$$

$$IH_2 = \frac{(1 + i)^2 - (1 + h)^2}{(i - h)} = \frac{2(i - h) + (i^2 - h^2)}{i - h} = 2 + i + h$$

$$IH_3 = \frac{(1 + i)^3 - (1 + h)^3}{(i - h)} = \frac{3(i - h) + 3(i^2 - h^2) + (i^3 - h^3)}{i - h}$$

$$= 3 + 3i + 3h + i^2 + ih + h^2$$

Those for $IH_j$ can be derived from the above and from the identity in equation (11):

$$IH_0 = \frac{0 - 0}{h - j} = 0$$

$$IH_1 = \frac{1 - 1}{h - j} = 0$$

$$IH_2 = \frac{(2 + i + h) - (2 + i + j)}{h - j} = 1$$

$$IH_3 = \frac{(3 + 3i + 3h + i^2 + ih + h^2) - (3 + 3i + 3j + i^2 + ij + j^2)}{h - j}$$

$$= 3 + i + h + j$$
Some Properties of the Accumulation Function

Definition

\[
\frac{R_1}{n} = (1 + r_1)^n
\]

\[
\frac{R_1 R_2}{n} = \sum_{t=1}^{n} R_{t-1} (1 + r_2)^{n-t}
\]

\[
\frac{R_1 R_2 \cdots R_m}{n} = \sum_{t=1}^{n} R_{t-1} (1 + r_m)^{n-t}
\]

Formula

\[
\frac{R_1 R_2 \cdots R_m}{n} = (r_1 - r_2)(r_1 - r_3) \cdots (r_1 - r_m) + (r_2 - r_1)(r_2 - r_3) \cdots (r_2 - r_m) + \cdots
\]

\[= \frac{n(n-1) \cdots (n-m+2)}{(m-1)!} (1 + r)^{n-m+1}
\]

if \( r = r_1 = r_2 = \cdots = r_m \)

Properties

(i) \( \frac{R_1}{R_2} = \frac{R_1 R_2}{R_2} = \frac{R_1 R_2 R_3}{R_3} = \cdots = 1 \)

\( \frac{R_1}{R_1} = \frac{R_1 R_2}{R_2} = \frac{R_1 R_2 R_3}{R_3} = \cdots = 0 \)

Lower powers are also equal to zero.
Properties (cont.)

(ii) Let a period (.) be used as an operating symbol such that, for example:

\[ R_1 R_2 \cdot R_3 R_4 = R_1 R_2 R_3 R_4 \]

Then \( R_1 R_2 \) may always be replaced by \( \frac{R_1 - R_2}{r_1 - r_2} \) and normal distributive rules apply.

For example:

(a) \[ \frac{R_1 R_2 R_3}{R_1 R_2} = \frac{R_1 R_2}{R_3} \]

\[ \frac{R_1 - R_2}{(r_1 - r_2)} \cdot R_3 \]

\[ \frac{R_1 R_3 - R_2 R_3}{(r_1 - r_2)} \]

(b) \[ \frac{R_1 R_2 R_3 R_4}{R_1 R_2} = \frac{R_1 R_2}{R_3 R_4} \]

\[ \frac{(\frac{R_1}{r_1 - r_2} - \frac{R_2}{r_3 - r_4})}{(\frac{R_3}{r_1 - r_2} - \frac{R_4}{r_3 - r_4})} \cdot \frac{1}{(r_1 - r_2)(r_3 - r_4)} \cdot (\frac{R_1 R_3}{R_1 R_3 - R_2 R_3 - R_1 R_4 + R_2 R_4}) \]
(iii) Define the increment in the function from \( n \) to \( n + 1 \) as follows:

\[
\frac{d}{dn} (R_1 R_2 \ldots R_m) = \frac{R_1 R_2 \ldots R_m}{n+1} - \frac{R_1 R_2 \ldots R_m}{n}
\]

Then:

\[
\frac{d}{dn} (R_1 R_2 \ldots R_m) - \frac{R_1 R_2 \ldots R_m}{n} = \frac{R_1 R_2 \ldots R_{m-1}}{n}
\]

\[
\frac{d}{dn} \frac{R_1}{n} - \frac{r_1}{n} \frac{R_1}{n} = 0
\]

Hence if \( dZ - r_m Z = a \frac{R_1 R_2 \ldots R_{m-1}}{n} \)

\[
Z = Z_0 \frac{R_m}{n} + a \frac{R_1 R_2 \ldots R_m}{n}
\]
ANNEX I

Notation, Brevity and Accuracy

As an example of the advantages of having a notation for the accumulation function at all, we might quote some formulas used by Chenery and Strout in their article on foreign assistance.¹ They distinguish a phase of development, when ability to invest is the constraint. In one variant the investment-saving gap determines foreign assistance. If investment is growing at a rate \( h \) (say),

\[
I = I_0 H
\]

Assuming no interest on external aid, as in the article, GNP or GDP is given by:

\[
Z = Z_0 + \frac{I_0 \ OH}{k}
\]

From this it is easy to deduce that:

\[
S = S_0 + j I_0 \ OH
\]

Since

\[
I = I_0 + h I_0 \ OH
\]

\[
F = F_0 + (h-j) I_0 \ OH
\]

Furthermore, if one wishes to determine the sum of aid over time, we merely have to accumulate the last equation at zero growth rate:

\[
\sum_{0}^{n} F = (n+1) F_0 + (h-j) \frac{I_0 \ OH}{n+1}
\]

The corresponding formulae as given in the article, but with symbols changed to make them conform to the usage here, are:

\[
\frac{F_n}{Z_0} = \frac{I_0}{Z_0} (1 + h)^n - s - \frac{I_0}{Z_0} \frac{s}{kh} \left( (1 + h)^n - 1 \right).
\]

\[
\frac{n}{Z_0} = \frac{F_n}{Z_0} \sqrt{(1 + h)^{n+1} - 1} \left( \frac{1 - s}{kh} - (n + 1) \left( s - \frac{I_0}{Z_0} \frac{s}{kh} \right) \right).
\]

The latter is the simpler of two versions given. The reader might be excused for being a little shy of it.

As a second example, we may cite a study by Qayum \(^1\) of the effects of a single loan on savings and income. In the first part of this study, it is assumed that the loan is repaid in equal annual instalments without a grace period and interest is paid on the outstanding balance. The savings function is implicitly assumed to be on a domestic savings basis. A calculation of the aggregate savings over the life of the loan is made in order to see whether this aggregate is positive or negative. Since aggregate savings determine the income ascribable to the loan in the year following the final instalment on it, this is the same as asking whether income in that year is positive or negative.

This question can be answered fairly simply. From Table 8, (page 35), we know that the indirect addition to GNP from a single investment of ka\(^1\) in year \((n + 1)\) is:

\[a'(j' - 1)OJ'.\]

---

However, repayment of the loan in equal annual instalments over a period of \( n \) years means that there is a corresponding series of negative investments, each of \( \frac{ka'}{n} \). Each of these yields a terminal indirect addition income in year \((n + 1)\). When added together, they equal:

\[-\frac{a'}{n} (j' - i)O0J'\]

Since the loan is repaid, the total external investment is zero and the direct additions cancel out. The net effect on income is therefore:

\[Z_{n+1} = a'(j' - i) (OJ' - O0J')\]

The second of these expressions is necessarily positive since \( O0J' \) is the sum of \( OJ' \) from \( t = 0 \) to \( n - 1 \). The sign of \( Z_{n+1} \), therefore, turns on whether \( j' \) is greater or less than \( i \). The period of repayment is irrelevant.

Qayum does not give a complete expression for the aggregate savings. He states the savings in each year are as follows (the symbols have again been changed to conform to usage here):

\[S_t = ka \sqrt{j' - (1 - \frac{t-1}{n})i} - \frac{1}{j'}\]

Aggregate savings are then expressed as follows:

\[S^* = ka + \sum_{t=1}^{n} S_t \frac{(1 - j'_{n-t+1})}{1 - j'}\]
This is not only far more complicated. It is also quite incorrect 1/. This statement is not made in a spirit of triumphant discovery of error; the writer has discovered too many similar ones of his own for that. The moral is, rather, that a simplified notational system can make life easier for both writer and reader.

1/ The error which Qayum makes is as follows. He states that the savings $S_t$ in year $t$ lead to additional savings in year $(t + 1)$ of $j'S_t$ and in year $(t + 2)$ of $j'^2S_t$ and so on. The first part of this is correct. The income in year $(t + 1)$ is $S_t/k$ and savings are $j'S_t$. But income in year $(t + 2)$ will be $S_t (1 + j')$ and savings correspondingly $j'S_t(1 + j')$ and not $j'^2S_t$. These savings, which may be positive or negative, are greatly understated.
In this Annex we shall develop an expression for GNP, when the income resulting from an investment starts not in the year following the investment, but in the subsequent year. In other words, there is a two-year lag. It will be assumed that there is zero interest.

Let us first consider an economy without external aid and with a constant savings rate. Previously, with a one-year lag, we had a relationship such that:

$$kdZ_n = I_n = sZ_n$$

or

$$dZ_n = jZ_n$$

But now it is the increment in income from year \((n + 1)\) to \((n + 2)\) which depends on savings in year \(n\). Therefore:

$$dZ_{n+1} = jZ_n$$

The solution to this is of the form:

$$Z_n = A q^n$$

where

$$q(1 + q) = j$$

This equation has two roots, which we shall call \(q'\) and \(q''\). One of them, say \(q'\), is slightly less than \(j\). The other, \(q''\), is related to \(q'\) in the following ways:

$$q' + q'' = -1$$

$$q' q'' = -j$$
The full expression for $Z_n$ thus includes both $q'$ and $q''$ and is of the form:

$$Z_n = A'Q'_{n} + A''Q''_{n}$$

The constants, $A'$ and $A''$, depend on both $Z_0$ and $Z_1$ because of the lagged relationship between investment and income. Specifically, it can be shown that:

$$Z_n = Z_1Q'Q''_{n} + Z_0jQ'Q''_{n-1}$$

Now $(1 + q'') = -q'$ which is small, so that, for any but the smallest values of $n$, $Q''_{n}$ is negligible. Hence, for values of $n$ over 5 (say), we can approximate as follows:

$$Z_n = \frac{Z_1Q'Q''_{n}}{q'-q''} + \frac{Z_0jQ'Q''_{n-1}}{q'-q''}$$

$$= \frac{Q'_{n}}{1+2q'}\sqrt{Z_1 + Z_0q'}$$

This means that the economy tends to grow at a rate $q'$. If, in fact, we assume that $Z_1 = (1 + q')Z_0$, then we can simplify still further and say that:

$$Z_n = Z_0Q'_{n}$$

The effect, then, is to reduce the rate of growth a little. With a three year lag, it would be reduced a little more. The corresponding values of $j$ and $q'$ in a few cases would be:

<table>
<thead>
<tr>
<th>$q'$</th>
<th>$j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>0.016</td>
</tr>
<tr>
<td>0.05</td>
<td>0.025</td>
</tr>
<tr>
<td>0.06</td>
<td>0.036</td>
</tr>
</tbody>
</table>
The lag, in sum, increases the effective capital-output ratio. Whether it is, in practice, worth observing the relationship between investment and income with a lag and then adjusting for it is seriously open to question.

External assistance complicates the problem considerably. If there is external assistance in year \( t \) of \( k a R_t \), the income from this investment will be \( a R_t \) in year \( (t + 2) \). In the subsequent \( n - (t + 2) \) years (i.e. up to and including year \( n \)), this will grow to:

\[
Z = a R_t \frac{Q^1 Q''}{n-t-1}
\]

As a result of the series of investments \( k a R_t \) from \( t = 0 \) to \( t = n-2 \), income will equal:

\[
Z_n = \sum_{t=0}^{n-2} a R_t \frac{Q^1 Q''}{(n-t-1)}
\]

\[
= \sum_{t=1}^{n-1} a R_{t-1} \frac{Q^1 Q''}{(n-t)}
\]

When \( t = n \), \( \frac{Q^1 Q''}{(n-t)} = \frac{Q^1 Q''}{0} = 0 \), therefore:

\[
Z = \sum_{t=1}^{n} a R_{t-1} \frac{Q^1 Q''}{(n-t)}
\]

\[
= a Q^1 Q'' R_n
\]

Again this can be approximated by omitting \( Q''_n \):

\[
Z_n = a \left[ \frac{Q^1}{(q^1-r)(q^1-q''')} + \frac{R_n}{(r-q')(r-q'')} \right] \]

\[
= \frac{a}{(q^1-r)} \left[ \frac{Q^1}{(1 + 2q')} - \frac{R_n}{(1 + q' + r)} \right] \]
The full expression for $Z_n$ can be obtained by combining the two parts:

$$Z_n = Z_1 Q^1 Q_n + Z_0 Q^1 Q_{n-1} + aQ^1 Q_{P_n}$$