A THEORY OF INTERLINKED RURAL TRANSACTIONS*

by

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A THEORY OF INTERLINKED RURAL TRANSACTIONS

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Abstract

This paper argues that a landlord/owner-cultivator who is unable
to monitor a tenant/laborer's effort input will link a wage-cum-output
sharing contract with the provision of consumption credit. The response of
contractual parameters to (a) alternative opportunities available to tenant/
laborers, (b) the cost of credit, and (c) the riskiness of cultivation is
then examined. The analysis shows that public policy designed to alleviate
rural poverty must recognize the delicate relationships between technological
considerations (prohibitive monitoring costs) and modes of transaction
(interlinked contracts) that arise in the absence of a complete set of
markets.
A THEORY OF INTERLINKED RURAL TRANSACTIONS

1. Introduction

Several writers on South Asian agriculture (Bharadwaj (1974), Bardhan and Rudra (1978), Jodha (1979)) have documented the widespread use of contracts which link labor, credit and output transactions among the same set of agents. Examples include the provision of credit by a landowner to tenants, wage laborers or farm servants and the leasing out of land to families relatively well endowed with nontraded factors such as bullocks and family labor. The rationale for the use of interlinked contracts may only be fully appreciated with reference to the wider social and political context within which the village economy is embedded. This paper, however, advances an economic explanation of the phenomenon which is rooted in the view that prevalent modes of transaction, of which interlinked contracts are an example, are shaped by technological considerations. Specifically, it is shown that interlinking is an efficient economic response to unequally distributed information arising from the uncertainty which characterizes subsistence agriculture. 1/

This has the implication that a situation where interlinking by landlords has been abolished and tenant/laborers allowed unrestricted access to official credit markets on competitive terms can be Pareto-inferior for rural economic agents. The analytical framework set up to demonstrate these results is then used to examine the response of the contractual parameters to changes in (1) the returns to alternative occupations available to tenant/laborers, (2) the cost of credit to the village economy and (3) the riskiness of cultivation.

1/ A valuable account of the importance of uncertainty in agricultural decision making is provided by Bliss and Stern (1979).
The rest of this introduction attempts to explain the approach taken in the paper and to interpret the results reported above. The use of interlinked contracts is not easily explained in economies characterized by a complete set of markets. Traditional subsistence agriculture is, however, subject to substantial risks which, in the presence of costly monitoring and moral hazard, preclude the functioning of such markets. Other institutions can then be expected to perform some of the roles economists usually assign to markets; it is argued below that interlinking is a response to such a need. The main point can be illustrated simply with an analogy. Costs of monitoring prevent a seller of car insurance from discovering to what extent an accident was due to bad luck or insufficient care. The probability of an accident can however be influenced by controlling transactions in related commodities so that insurers would like to see a tax on sales of alcoholic beverages to motorists. Likewise in agriculture. A landlord/owner-landlord may be unable to tell whether the observed low output of a particular tenant/laborer is due to adverse circumstances or inadequate effort. There is, however, a conventional externality imposed by a cultivator's borrowing on the amount of effort he chooses to expend; efficiency in allocation then requires intervention in the credit decision. This takes the form of interlinking credit and output contracts.

The above argument for interlinking is developed in the paper in the context of a principal-agent model. A group of identical tenants/laborers is brought together by a landlord/owner-landlord who maximizes profits subject to tenants getting no less than a reservation utility level. Production decisions are influenced via a linear incentive system, i.e. one where $y$, the amount produced by a farmer, is related to $z$, the amount
retained by him as follows: \( z = \alpha + \beta y; \alpha, \beta > 0 \). \( \alpha \) and \( \beta \) may naturally be regarded, following Stiglitz (1974) as the "pure wage" and output sharing parameter respectively. It can then be shown that Pareto efficiency requires exercising control over the amount of consumption credit made available to tenant/laborers.

Three sets of comparative statics exercises are performed and their impact on the contractual parameters ascertained. The first traces the consequences of varying the reservation utility level summarizing alternative opportunities available to tenant/laborers. The second examines the effects of changing the cost of credit to the system. The third explores the results of a mean-preserving change in the riskiness of cultivation.

The principal results for utility functions additively separable in present and future consumption and effort are given below.

Let \( \sigma \) denote the elasticity of a tenant/laborer's marginal utility of consumption. If the output-sharing parameter, \( \beta \), is fixed at a conventional norm (50% in parts of Indian agriculture), an increase in tenants' reservation utility level (1) cheapens the cost of credit provided by landlords, (2) reduces, leaves unchanged or increases the pure wage according to either \( \sigma \geq 1 \), and (3) causes more borrowing if \( \sigma \leq 1 \). A mean-preserving reduction in the riskiness of agricultural production (1) raises the cost of consumption credit and (2) raises the pure wage if \( \sigma \geq 1 \).

If the intermittently discussed policy of abolishing interlinking were implemented and tenants given unrestricted access to office credit markets, subsidization of that credit would (1) lower the tenant's cropshare
if \( \sigma < 1 \); (2) lower the pure wage if \( \sigma \geq 1 \); and (3) reduce the amount of borrowing if \( \sigma = 1 \).

An important implication of the analysis developed in this paper is that public policy designed to help the rural poor must duly recognize the relationships between technological considerations (prohibitive monitoring costs) and modes of transaction (interlinked contracts) to which they give rise. Thus, well-meaning attempts to abolish moneylending by landlords and to grant tenants access of official credit markets at competitive rates would actually be Pareto-worsening for landlords and tenants. This result, which is rather striking, may be related to the line of work developed by Mirrlees (1974), Hart (1975), Diamond and Mirrlees (1978), and Newbery and Stiglitz (1979b). Those authors showed that the opening of a market which adds to an incomplete set of contingent markets, without however completing them, may generate Pareto-inferior outcomes.

We close this introduction by mentioning related work on agrarian contracts. The principal-agent formulation of the landlord-tenant problem is developed with many interesting examples in Newbery and Stiglitz (1979a) and extended, most recently, to analyze interlinked contracts by Braverman and Stiglitz (1982). That model is used by Braverman and Srinivasan (1981) to examine sharecropping-cum-credit contracts in a village economy without uncertainty. Problems of coordination between landlord and moneylender are also explored in Bell and Zusman (1980).

The plan of this paper is as follows. Section II outlines the principal-agent model and derives the properties of a Pareto-efficient allocation both with and without moral hazard. It then demonstrates that a moral hazard constrained allocation is in general not attainable without
interlinked contracts. Section III is devoted to comparative statics analysis. Section IV concludes the paper.

II. Interlinked Contracts

Farmers

Consider a community of identical landless farmers/wage laborers who deal with a single landlord/employer. A typical farmer enters into a contract with the landlord/employer to work in period 1 in return for a wage, $a$, and a share, $\beta$, of the output he produces. The harvest, however, becomes available only in period 2, so that the farmer must borrow to finance first period consumption. The loan is paid back, with interest, in the second period.

Let

\[ c_i \] consumption in period $i$ ($i = 1, 2$)

\[ L \] labour time

\[ e \] effort ($0 \leq e \leq 1$).

It is assumed that effective labor input equals $eL$. All farmers work the same number of hours, so that $L$ may be chosen to equal unity. An individual's utility is represented by a concave function $u(c_1, c_2, e)$. Use the notation $u_i$ to signify the partial derivative with respect to argument $i$. Then $u_1, u_2 > 0$, $u_3 < 0$. It is also assumed that $u_1 \to \infty$ as $c_i \to 0$ ($i = 1, 2$) and $u_3 \to \infty$ as $e \to 1$. Thus the marginal utility of consumption and the marginal disutility of effort become infinite at zero consumption and maximum effort respectively.

An individual farmer's production possibility is subject to uncertainty-output, $Y$, depends on a random variable, $\Theta$, representing the
state of nature and on effective labour, \( e \), which must be applied (in period 1) before the state of nature is known (in period 2)

\[ Y = f(e, \theta) \quad (1) \]

\( f \) is increasing and strictly concave in \( e \), i.e. \( f'_e > 0 \), \( f_{ee} < 0 \). The variable \( \theta \) can assume a number of values \( \theta_j \) with known probabilities \( p_j \) \((j = 1, \ldots, n)\). To highlight the importance of uncertainty, it will be assumed that states of nature, \( k \), where \( f(e, \theta_k) = 0 \), all \( e \), may occur with non-zero probability. Relation (1) does not contain any explicit reference to land; its inclusion would not add much to the analysis presented here.

The Landlord

The profit accruing to a landlord in the \( j \)th state of nature equals the difference between production in that state of nature and the sum of wages and tenant/labourers' share of the crop. If the landlord has access to a perfect organized capital market at a rate of interest, \( i \), his present value profit in the \( j \)th state of nature, \( T_j \), may be written

\[ T_j = r[(1-\beta)f(e, \theta_j) - \alpha] \quad (j = 1, \ldots, n) \quad (2) \]

where \( r = \frac{1}{1+i} \).

Since \( \alpha \) and \( \beta \) are agreed upon in advance of knowing which state of nature will occur, it is clear from (2) that \( T_j \) could be negative in states of nature where output \( f \) becomes zero. It is assumed that the landlord has other sources of income large enough to absorb those losses. Under these circumstances, the landlord is likely to be considerably less risk-averse.
than his tenant/laborers. We assume that he is risk-neutral and therefore that his objective is the maximization of expected profits, \( E \).

The landlord's task is to choose a risk-sharing scheme, here described by \( \alpha \) and \( \beta \) and hence restricted to be linear, and other control variables to specified to maximize expected profit

\[
E[(1-\beta)f(e, \theta) - \alpha]
\]

subject to

\[
E[u(c, \alpha + \beta f(e, \theta) - c(1+i), \theta)] > u
\]

where (2) has been used in writing (3). (4) states that the expected utility of a tenant/laborer must not fall short of \( u \), a reservation utility level summarizing opportunities available elsewhere. Notice that second period consumption is the sum of wages and share of output less repayment of the consumption loan. It is assumed that tenants may borrow at the rate \( i \) as well; it will be shown that even with this assumption about access to the capital market on equal terms, efficiency in allocation in the presence of moral hazard will call for measures to influence a tenant's consumption loan decision. In the rest of the paper we shall formulate and analyze a sequence of models where the landlord can exercise progressively less control over individual farmers' decisions.

**Unconstrained Pareto Efficiency**

A useful benchmark for subsequent analysis is provided by an allocation where the landlord can directly control (a) the application of effort and (b) the amount of consumption in both periods, subject only to
ensuring a tenant/laborer a utility level no smaller than \( \bar{u} \). The landlord's problem is therefore one of choosing \( c, e, \alpha \) and \( \beta \) to maximize (3) subject to (4). It will be established that the solution to this is Pareto efficient, i.e., that it does not permit an increase in ET (respectively Eu) without reducing or leaving unchanged Eu (respectively ET).

To this end, we first provide an intuitive argument to show that (4) will hold as an equality at a solution to the landlord's problem. For if not, it is always possible to reduce the pure wage \( \alpha \) slightly and keep \( c, e \) and \( \beta \) unchanged without violating the constraint (4). Since a feasible reduction in the pure wage increases the landlord's profit (3), the original situation could not have been optimal. We have therefore motivated

**Proposition 1:** The reservation utility constraint (4) holds as an equality at a solution to the landlord's maximization of (3) subject to (4) with respect to the variables \( c, e, \alpha, \beta \).

**Proof:** Since \( u_2 \to \infty \) as a second period consumption tends to zero, and there is a nonzero probability of states of nature where output is zero no matter how much effort is applied, \( \alpha \) must be positive at any solution to the maximization problem. The proof proceeds by contradiction. Suppose that (4) is a strict inequality at a solution to the landlord's maximization. Denote that solution by \( (c^*, e^*, \alpha^*, \beta^*) \). Since \( \alpha^* > 0 \) and the utility function of tenants is continuous, there exists an arbitrarily small positive number \( \varepsilon \), such that \( \alpha_0 = \alpha^* - \varepsilon \) is positive and continues to satisfy (4). The effect of such a feasible reduction in \( \alpha \) is to increase expected profit in (3), i.e., \( ET(c^*, e^*, \alpha_0, \beta^*) > ET(c^*, e^*, \alpha^*, \beta^*) \). But this contra-
dicts the supposed optimality of the starred allocation. Hence (4) must hold as an equality.

The argument that (4) always holds with equality, together with the concavity of the utility function, implies that there exists a positive number, \( \lambda \), such that a solution to the landlord's problem \((c^*, e^*, \alpha^*, \beta^*)\) maximizes

\[
r E[(1-\beta)f(e, \theta) - \alpha] + \lambda E u
\]

(5)

where \( \lambda \) may be interpreted as the shadow price of (4) in terms of the landlord's expected profit. But this implies that the starred allocation is Pareto efficient. For if there were another feasible allocation (denoted by bars) with the property that \((E_T, \bar{E}_u) \geq (E_T^*, \bar{E}_u^*)\) with strict inequality in at least one component, it would lead to a higher value of (5), a contradiction. This establishes

**Proposition 2:** A solution to the expected profit maximization problem with respect to \( c, e, \alpha, \beta \), subject to a reservation utility constraint is Pareto efficient.

We next derive some other properties of a solution to (3) subject to (4). With (5) as the Lagrangean, the first-order conditions with respect to \( c, e, \alpha, \beta \) are

\[
E u_1 \leq \frac{1}{r} E u_2 \quad (c \geq 0)
\]

(6)

(on remembering that \( r = \frac{1}{1+r} \))

\[
r(1-\beta)E f_e + \lambda E (u_e^2 f_e + u_2) \leq 0 \quad (e \geq 0)
\]

(7)

\[-r + \lambda E u_2 \leq 0 \quad (\alpha \geq 0)
\]

(8)

\[-rE f + \lambda E u_2 f \leq 0 \quad (\beta \geq 0)
\]

(9)
where each inequality bears the relation of complementary slackness with the variable in brackets appearing on the right.

Since the landlord is risk neutral, and "disasters" a real possibility, we should expect him to bear all the risk in any Pareto efficient allocation. This leads to

Proposition 3: A Pareto efficient allocation is characterized by pure wage contracts alone, i.e. \( \beta = 0 \).

Proof: Let

\[ c_{2j} : \text{a farmer's second period consumption in the } j^{\text{th}} \]

\[ \text{state of nature.} \]

The linear risk sharing scheme makes

\[ c_{2j} = \alpha + \beta f(e, \theta_j) - c(l+i) \tag{10} \]

where the pure wage component, \( \alpha \), is certain but \( \beta f(e, \theta_j) \) is not.

Suppose, contrary to the proposition, that \( \beta > 0 \). It then follows from (10) that \( \text{Cov}[c_{2j}, f(e, \theta_j)] > 0 \) where \( \text{Cov} \) denotes covariance. Let

\[ u_{2j} : \text{a farmer's marginal utility of consumption in the } j^{\text{th}} \]

\[ \text{state of nature.} \]

By definition,

\[ u_{2j} = u_2(c, c_{2j}) . \]

Concavity of the utility function ensures that \( u_2 \) is nonincreasing in \( c_{2j} \) which, combined with \( \text{Cov}[c_{2j}, f(e, \theta_j)] \) implies \( \text{Cov}[u_2, f(e, \theta_j)] < 0 \).

To summarize the argument so far,

\[ \beta > 0 \text{ implies } \text{Cov}[u_2, f(e, \theta_j)] < 0 . \tag{11} \]
Since $\alpha > 0$ at a solution, (8) must hold as an equality. From (8) and (9) it then follows that

$$Eu_2^f \leq Eu_2 \mbox{ Ef (} \beta \geq 0$$

i.e., by definition, $\mbox{Cov}[u_2^f(e, \Theta_j)] \leq 0 \mbox{ (} \beta \geq 0\mbox{)}$.

This relation implies that

$$\beta > 0 \mbox{ implies } \mbox{Cov}[u_2^f(e, \Theta_j)] = 0 \hspace{1cm} (12)$$

But (12) contradicts (11) so that $\beta = 0$ at a Pareto efficient allocation.

The use of pure wage contracts implies from (10) that second period consumption, $c^2_j = \alpha - c(l+i)$, i.e., a constant independent of the state of nature. This result indicates that whether or not the landlord cultivates his land with pure wage labor is endogenous to the problem analyzed here. We therefore have

**Proposition 4:** A Pareto efficient allocation equates consumption across all states of nature.

The complete insurance afforded a farmer under such an arrangement leaves him no incentive to work. Since effort confers disutility, the allocation is unattainable without the sort of centralized labor direction we have assumed. We next examine a second-best allocation where farmers are free to make their own effort supply decisions.

**Second-Best Pareto Efficiency**

The second-best problem considered here arises because the landlord must respect the sovereignty of farmers' decentralized effort supply decisions. However, he continues as before to control their consumption decisions. This is in keeping with our strategy of analyzing models where
the landlord exercises a progressively diminishing degree of control. The consequences of allowing farmers to take their own effort supply and consumption decisions is central to the paper and will be examined at some length below.

Farmers

Farmers, who are assumed to be expected utility maximizers, select their effort supply in period 1 before knowing which state of nature will occur; the state of nature is revealed to them in period 2. A farmer's problem is to choose $e$ to maximize

$$W = Eu[c, a + \beta f(e, \theta) - c(1+i), e]$$

which leads to the first order condition

$$\frac{W_e}{e} = \beta Eu_{e2} + Eu_3 = 0$$

(13)

For future reference, it is useful to note that

$$W_{ee} = \beta E[u_{21} - (1+i)u_{22}] f_e + E[u_{31} - (1+i)u_{32}]$$

where the notation $u_{ij}$ denotes cross partial derivatives. The effect of borrowing on effort is given by

$$\frac{de}{dc} = -\frac{W_{ec}}{W_{ee}}$$

Since $W_{ee} < 0$, by second order conditions,

$$\text{Sign } \frac{de}{dc} = \text{Sign } \{\beta E[u_{21} - (1+i)u_{22}] f_e + E[u_{31} - (1+i)u_{32}]\}$$

(14)
Similarly,

$$W_e = \beta E_{u_2} e + E_{u_2}$$

and

$$\text{Sign} \frac{de}{da} = \text{Sign} \{\beta E_{u_2} e + E_{u_2}\}$$

(15)

The Landlord

Monitoring costs are assumed prohibitive and preclude the landlord's ascertaining (a) the true state of nature on a farm both before and after the event and (b) the amount of effective labor (recall $L=1$) applied by a farmer. He can only observe every farmer's output and is therefore unable to decide, for example, whether a case of low output is due to adverse circumstances or inadequate effort. This is a reasonable assumption: the pace, thoroughness, efficiency and inventiveness of the agent, subsumed in the variable $e$, have been identified by Stiglitz (1974) and other writers on contractual relationships as being extremely expensive to monitor. Output therefore serves as an easily measurable but imperfect surrogate for what the landlord would really wish to know, viz., effective labor input. The contractual parameters $c, \sigma$ and $\beta$, being based on what is observable by the landlord, are not state-dependent. Notice that the monitoring problem arises even if the landlord is dealing with one tenant/laborer. The assumption of a number of tenant/laborers who are identical ex ante but, because of the operation of the random element, non-identical ex post does not affect the severity of the monitoring problem.
The landlord's problem is to choose \( c, \alpha \) and \( \beta \) to maximize

\[
R = rE[(1-\beta)f(e(c,\alpha,\beta), \theta) - \alpha] \tag{16}
\]

subject to

\[
V = Eu[c, \alpha + \beta f(e(c,\alpha,\beta), \theta) - c(1+1), e(c,\alpha,\beta)] \geq \bar{u} \tag{17}
\]

In order to show that a solution to the above problem is constrained Pareto efficient, it is necessary to argue as before that, under certain assumptions, (17) will hold as an equality. We therefore establish

**Proposition 5:** The reservation utility constraint (17) holds as an equality at a solution to the landlord's maximization of (16) subject to (17) with respect to the variables \( c, \alpha, \beta \), provided that \( u_{32} \leq 0 \).

**Proof:** Denote a solution to the above problem by \((c^*, \alpha^*, \beta^*)\). Suppose that (17) is a strict inequality at that solution. It has been established before that \( \alpha^* > 0 \). Since the utility function and effort supply function are continuous, there exists an arbitrarily small positive number \( \varepsilon \), such that \( \alpha_0 = \alpha^* - \varepsilon \) is positive and continues to satisfy (17); its effect on \( Eu \) via the effort supply function \( e(c,\alpha,\beta) \) may be ignored because of the envelope theorem. The effect of such a feasible reduction in \( \alpha \) on expected profit is given, from (16), by

\[
R(c^*, \alpha_0, \beta^*) - R(c^*, \alpha^*, \beta^*) = -\varepsilon \eta [(1-\beta)EF_e \frac{de}{d\alpha} - 1] \tag{18}
\]

where the derivatives appearing on the right hand side of (18) are evaluated at \((c^*, \alpha^*, \beta^*)\). From (15). \( \frac{de}{d\alpha} > 0 \) if \( u_{22} \leq 0 \) and \( R(c^*, \alpha_0, \beta^*) \) is a superior feasible allocation, contradicting the supposed optimality of \((c^*, \alpha^*, \beta^*)\). Hence (17) must hold as an equality provided that the marginal utility of second period consumption does not increase with effort.\(^2\)

\(^2\) A sufficiently rapid increase in the marginal utility of second period consumption with effort would lead to a large reduction in effort as \( \alpha \) is decreased. This would prevent the landlord from using \( \alpha \) as an instrument to push a tenant/laborer to his reservation utility level.
An argument analogous to that preceding Proposition 2 now establishes

Proposition 6: A solution to the expected profit maximization problem with respect to $c, \alpha, \beta$ subject to a reservation utility floor is Pareto efficient constrained by tenant's choice of effort supply, provided $u_{32} \leq 0$.

A constrained Pareto efficient allocation is one where the landlord, although respecting maximizing farmers' effort supply decisions, continues to control the amount of consumption made available in the first period. This implies, of course, that individual farmers are not permitted to borrow and consume in accordance with their own wishes, a notably stringent requirement. We demonstrate that a second best allocation is characterized by the landlord's prohibiting farmers unrestricted access to the organized credit market.

The Lagrangean corresponding to the maximization of (16) subject to (17) is

$$\lambda = rE[(1-\beta)f(e(c, \alpha, \beta), \theta) - a] + \lambda(V-\bar{V})$$

where, from (17), $V = EU'[c, \alpha + \beta f(e(c, \alpha, \beta), \theta) - c(1+1), e(c, \alpha, \beta)]$. On using the envelope theorem, the first order condition with respect to $c$, the consumption loan size, is

$$r(1-\beta) \frac{de}{dc} Ef_c + \lambda V_c = 0 \quad (18)$$

Since $\lambda$ measures the increase in expected profit following a reduction in $\bar{V}$, it is positive. Hence

$$V_c \lesssim 0 \quad \text{according as} \quad \frac{de}{dc} \lesssim 0$$

$$V_c \gtrsim 0 \quad \text{according as} \quad \frac{de}{dc} \gtrsim 0$$

$$V_c = 0 \quad \text{according as} \quad \frac{de}{dc} = 0$$

$$V_c = \text{constant} \quad \text{according as} \quad \frac{de}{dc} = \text{constant}$$
A farmer, if allowed to choose $c$ freely, would set $V_c = 0$. From (19), this is satisfied at a second best Pareto efficient allocation only when $\frac{\partial e}{\partial c} = 0$, a condition that cannot generally be expected to hold. The landlord therefore uses the loan sign as an instrument to influence a farmer's (unobservable) effort level. In words, (19) states that at a second best, a farmer would wish to borrow less (respectively more) depending on whether effort increases (respectively decreases) with borrowing. From (14) and (19),

$$V_c \leq 0 \quad \text{according as} \quad (\beta \text{E}[u_{21} - (1+i)u_{22}] + \text{E}[u_{31} - (1+i)u_{32}]) \leq 0$$

(20)

For intertemporally additive utility, i.e., for

$$u = \phi(c,e) + \text{E} \psi(a + \beta f - c(1+i)),$$

(20) reduces to

$$V_c \leq 0 \quad \text{according as} \quad \phi_{ce} \geq \beta (1+i) \text{E} \psi'' f_e$$

(21)

where single and double primes respectively denote the first and second derivatives of single-variable functions. We therefore have

Proposition 7: With au intertemporally additive utility function, a tenant/laborer would wish to borrow less (respectively more) than at a second best Pareto efficient allocation according to whether

$$\phi_{ce} \geq \beta (1+i) \text{E} \psi'' f_e.$$

3/ All derivatives are evaluated at the second best solution.
Corollary: In the special case of additive separability \( \phi_{ce} = 0 \), a tenant/laborer would wish to borrow less than at a second best Pareto efficient allocation.

The above discussion shows that (constrained) Pareto efficiency requires that the consumption-leisure tradeoff be influenced by restricting farmers' access to the capital market. In this model, farmers may borrow at the same rate, i, as the landlord; it is nevertheless efficient to control the amount borrowed at that rate directly.

These results suggest that "delinking" of credit contracts from output contracts would lead to a reduction in profits without any change in tenant welfare. This is formally recorded in

**Proposition 8:** When the utility function is intertemporally additive and \( \phi_{ce} > 0 \), an interlinked credit-cum-output contract cannot be Pareto dominated by a delinked contract where farmers are permitted to make borrowing and effective labour decisions freely.

**Proof:** A tenant/laborer's problem is one of jointly choosing \( c \) and \( e \) to maximize

\[
Eu[c, a + \beta f(e, \theta) - c(i+1), e] .
\]

It can be shown (see Appendix 1) that \( \frac{de}{da} < 0 \) provided \( \phi_{ce} > 0 \).

The landlord's task is to choose \( \alpha \) and \( \beta \) to maximize

\[
rE[(1-\beta)E[e(\alpha, \theta), \theta] - \alpha]\]

subject to

\[
Eu[c(\alpha, \beta), \alpha + \beta f(e(\alpha, \beta), \theta) - c(\alpha, 3)(1+i), e(\alpha, \beta)] \geq \bar{u}
\]
An argument similar to that used to establish Proposition 5 (with the envelope theorem applying to both \( c(\alpha, \beta) \) and \( e(\alpha, \beta) \)) then shows that (23) will hold as an equality. A tenant/laborer is thus no better off than at a second-best Pareto efficient allocation. The maximum value of (22) cannot be greater and will in general be less than that attainable in the second-best problem. This is because the landlord must make do with controlling \( \alpha \) and \( \beta \) rather than the triple \( (\alpha, \beta, c) \). This establishes Proposition 8.

We have therefore provided an economic argument which helps explain why landlords can impose significant restrictions on farmers' freedom to operate on other markets. Such restrictions can then be sanctioned by social custom and implemented through the use of extra economic coercion. But to help appreciate the economics of the argument better, consider a somewhat different example. Suppose that a group of landless farmers is brought together by a cooperative authority which arranges production and distribution to maximize the collective welfare of its constituents, subject to the need to pay landowners a minimum rent for leasing out land. Since this formulation of the problem interchanges the objective and constraint in (16) and (17), it follows from Proposition 7 that the cooperative authority will link credit and output contracts in the interests of its own members. The argument for interlinking is therefore based on efficiency and not on the institutional circumstances which assign principal and agent roles to particular individuals. This has the important implication that public policy designed to help the rural poor must recognize the relationships between technological considerations (prohibitive monitoring costs) and modes of transaction (interlinked contracts) that arise in the absence of a complete set of markets.
III. Comparative Statics

This section specializes the model to ascertain the response of contractual parameters to changes in underlying data describing the economy. Specifically, it is assumed

(1) that there are two states of nature $\Theta_1$ and $\Theta_2$ occurring with probabilities $p$ and $(1-p)$ respectively;
(2) that effort $e$ can take two values 0 and 1 (as before $L=1$);
(3) that the production function $y = f(e, \Theta)$ has the following properties:

\[
\begin{align*}
    f(\Theta_1, 1) &= Q \\
    f(\Theta_2, e) &= 0 \quad \text{for all } e \\
    f(0, \Theta) &= 0 \quad \text{for all } \Theta
\end{align*}
\]  

Thus the cause of zero output can either be bad luck or lack of effort.

(4) for any pair $(c_1, c_2)$, the utility function where $e = 1$ ($u^1(c_1, c_2)$) and where $e = 0$ ($u^0(c_1, c_2)$) are related as follows:

\[
    u^1(c_1, c_2) = u^0(c_1, c_2) - A
\]  

where $A > 0$.

The second best Pareto efficient allocation with control over consumption credit and an allocation with completely unrestricted access to organized credit may both be regarded as polar cases which serve to organize our thoughts on this subject. An intermediate formulation which allows the landlord partial control is to suppose that he can set a price at which agents undertake utility-maximizing borrowing. Let $t$ denote an interest tax or subsidy charged to agents, so that $t$ is the rate of interest at which they may borrow in the first period.

\[4/\] The model is an adaptation of that introduced by Diamond and Mirrlees (1978) to analyze social insurance.
If the utility function is assumed intertemporally additive, the expected utility from applying effort is
\[
\phi(c^1) + p\phi[a+cQ-c^1(1+ti)] + (1-p)\phi[a-c^1(1+ti)] - A
\]
while the expected utility from not applying effort is
\[
\phi(c^0) + \phi[a-c^0(1+ti)]
\]
where
\[
c^1: \text{optimal borrowing when } e = 1
\]
\[
c^0: \text{optimal borrowing when } e = 0^5/
\]
To ensure that farmers choose \( e = 1 \), it is necessary for the landlord to ensure that (26) is at least as large as (27). To simplify the algebra it will be assumed that borrowing decisions are made on the premise that \( e = 1 \), so that the constraint may be written
\[
\phi(c) + p\phi[a+cQ-c(1+ti)] + (1-p)\phi[a-c(1+ti)] - A \geq \phi(c) + \phi[a-c(1+ti)]
\]
where \( c \) is optimal borrowing when \( e = 1 \). This may be simplified to
\[
\phi[a+cQ-c(1+ti)] - \phi[a-c(1+ti)] - \frac{A}{p} \geq 0
\]
(28) is sometimes referred to as the moral hazard constraint.

With these assumptions, the landlord's problem is one of choosing \( \alpha, \beta \) and \( t \) to maximize
\[
F = \frac{1}{1+t}(p(1-\beta)Q-c(a,\beta,t)(1-t))
\]
(29)

\(^5/\) This formulation assumes that farmers do not default on their consumption loans when it suits them to do so.
subject to

$$H = \frac{1}{2}[\alpha + \beta Q - c(\alpha, \beta, t)(1 + ti)] - \frac{1}{2}[\alpha - c(\alpha, \beta, t)(1 + ti)] - \frac{A}{p} \geq 0 \quad (30)$$

$$R = \phi(c(\alpha, \beta, t)) + p\phi[\nu + \beta Q - c(\alpha, \beta, t)(1 + ti)] + (1 - p)\phi[\alpha - c(\alpha, \beta, t)(1 + ti)] - \bar{u} \geq 0 \quad (31)$$

where (30) is the moral hazard constraint and (31) ensures that expected utility when effort is applied is no smaller than the reservation utility level, $\bar{u}$.

To derive comparative statics results, form the Lagrangean

$$M = F + \eta H + \pi R \quad (32)$$

**Reservation Utility:** We first consider how the variables $\alpha$, $\beta$ and $t$ vary with $\bar{u}$, which summarises alternative opportunities available to tenant! laborers. This calls for a solution to the following system

$$
\begin{bmatrix}
M_{\alpha} & M_{\alpha \beta} & M_{\alpha t} & H_{\alpha} & R_{\alpha} \\
M_{\beta} & M_{\beta \beta} & M_{\beta t} & H_{\beta} & R_{\beta} \\
M_{t} & M_{t \beta} & M_{t t} & H_{t} & R_{t} \\
H_{\alpha} & H_{\beta} & H_{t} & 0 & 0 \\
R_{\alpha} & R_{\beta} & R_{t} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
da \\
d\beta \\
dt \\
d\eta \\
d\bar{u}
\end{bmatrix}
= 
\begin{bmatrix}
-M_{\alpha u} \\
-M_{\beta u} \\
-M_{t u} \\
-H_{u} \\
-R_{u}
\end{bmatrix}
\quad (33)
$$

Solutions to the complete system (33) are quite complicated. We have therefore found it useful to concentrate on a number of special cases derived by holding constant one of the variables $\beta$, $t$ and $\alpha$. 

The propositions which follow in this section arise from straightforward but tedious manipulation of equation systems like (33); proofs are sketched in Appendix 2.

(1) **Constant β**: Imagine a society where the cropshare β is conventionally fixed (e.g., at 50% in some parts of India). On deleting the β row and column in (33), we are able to derive

**Proposition 9**: With β held constant (i) \( \frac{dt}{du} < 0 \), (ii) \( \frac{da}{du} > 0 \) as \( \sigma > 1 \),

(iii) \( \frac{dc}{du} > 0 \) if \( \sigma < 1 \).

An increase in reservation utility causes landlords to cheapen consumption credit [part (i)]. A sufficiently rapid decline in the marginal utility of consumption (\( \sigma > 1 \)) however requires a cut in the pure wage \( a \) to maintain work incentives [part (ii)]. Borrowing unambiguously increases in response to cheaper credit if \( \sigma < 1 \) (part (iii)) in which case the pure wage goes up as well (part (ii)).

(2) **Constant t**: A special case of this is the situation where the institution of interlinking is successfully abolished and tenant/laborers are granted access to official credit at competitive rates of interest. The next proposition is however not confined to the case where \( t = 1 \).

**Proposition 10**: With t held constant (i) \( \frac{db}{du} > 0 \), (ii) \( \frac{da}{du} > 0 \),

(iii) \( \frac{dc}{du} > 0 \).

These results are intuitively satisfactory and require no explanation.

(3) **Constant a**: With an institutionally fixed pure wage, we are able to derive
Proposition 11: With $\alpha$ held constant (i) $\frac{d\theta}{d\theta} > 0$ if $\sigma < 1$,
(ii) $\frac{d\theta}{d\theta} < 0$ if $\sigma > 1$, (iii) $\frac{dc}{d\theta} > 0$ if $\sigma > 1$.

An increase in reservation utility is accompanied by measures to preserve incentives—increasing agents' cropshare and cheapening the cost of credit—when the marginal utility of consumption declines rapidly.

Cost of Credit: We next examine the consequences of changes in the cost of organized credit, $i$. The system to be solved is

\[
\begin{bmatrix}
M_{\alpha} & M_{\beta} & M_{\alpha t} & H_{\alpha} & R_{\alpha} & \vdots & -M_{\alpha i} \\
M_{\beta} & M_{\beta} & M_{\beta t} & H_{\beta} & R_{\beta} & \vdots & -M_{\beta i} \\
M_{t\alpha} & M_{t\beta} & M_{tt} & H_{t} & R_{t} & \vdots & -M_{ti} \\
H_{\alpha} & H_{\beta} & H_{t} & 0 & 0 & \vdots & -H_{i} \\
R_{\alpha} & R_{\beta} & R_{t} & 0 & 0 & \vdots & -R_{i}
\end{bmatrix}
\begin{bmatrix}
da \\
d\beta \\
dt \\
d\eta \\
d\pi \\
\end{bmatrix}
= 
\begin{bmatrix}
-M_{\alpha i} \\
-M_{\beta i} \\
-M_{ti} \\
-H_{i} \\
-R_{i}
\end{bmatrix}
\]

We may now establish

Proposition 12: With $\tau$ held constant (i) $\frac{d\beta}{d\tau} > 0$ if $\sigma < 1$,
(ii) $\frac{da}{d\tau} > 0$ if $\sigma > 1$, (iii) $\frac{dc}{d\tau} > 0$ if $\sigma = 1$.

This proposition shows that in a situation characterized by no interlinking, a cheapening of official rural credit will lead landlords to reduce the tenant/laborer's cropshare when $\sigma < 1$. It leads to a cut in the pure wage in the case where maintenance of work incentives is potentially important ($\sigma > 1$). The combination of the above leads to the following seemingly perverse result when $\sigma = 1$. Borrowing for consumption decreases when official credit is cheapened because of offsetting actions by landlords.
Riskiness of Cultivation: We conclude this investigation by examining the effects of a mean preserving change in the riskiness of cultivation. Since mean output equals $pQ$, it is postulated that $p$ and $Q$ change together while satisfying $d[pQ] = 0$, i.e.,

$$pdQ + Qdp = 0$$

(35)

The variance of the distribution is $p(1-p)Q^2$ and $\frac{d[p(1-p)Q^2]}{dp} = -Q^2 < 0$ when $d[pQ] = 0$. Hence $dp > (0), dQ < (>)0$ satisfying (35) corresponds to a mean preserving reduction (increase) in risk.

The effects of such a change may be determined by solving the following system:

$$
\begin{bmatrix}
M_{\alpha} & M_{\beta} & M_{tt} & H_{\alpha} & R_{\alpha} \\
M_{\beta} & M_{\beta} & M_{tt} & H_{\beta} & R_{\beta} \\
M_{t\alpha} & M_{t\beta} & M_{tt} & H_{t} & R_{t} \\
H_{\alpha} & H_{\beta} & H_{t} & 0 & 0 \\
R_{\alpha} & R_{\beta} & R_{t} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
da \\
d\beta \\
dt \\
dn \\
d\varphi
\end{bmatrix}
=
\begin{bmatrix}
-M_{ap} \\
-M_{bp} \\
-M_{tp} \\
-H_p \\
-R_p
\end{bmatrix}
$$

(36)

where $dp$ and $dQ$ are related as in (35).

Proposition 13: If the moral hazard constraint (30) is binding, then with fixed $\beta$ (i) $\frac{dt}{dp} > 0$, (ii) $\frac{d\alpha}{dp} > 0$ if $\sigma > 1$, provided $\phi'''' > 0$, where three primes denote a third derivative.
The assumption of \( \phi''' > 0 \) is plausible and is implied by the hypothesis of decreasing absolute risk aversion. With (30) binding and \( \alpha \) constant, a mean-preserving reduction in risk increases the utility of working relative to that from inactivity. This reduces the magnitude of the externality to which interlinking is a response. Recall that with intertemporally additive utility, agents wish to borrow less than the principal would like, thus establishing a presumption for subsidized credit.\(^6\) A reduction of interlinking is then associated with an increase in the cost of borrowing (part (i)). The need to preserve work incentives when \( \sigma \geq 1 \) leads to an increase in the pure wage (part (ii)).

It is worth noting that borrowing may go up or down depending on the particular form of the utility function. Furthermore, the landlord’s profit, \( F \), as defined in (29) may either increase or decrease following a mean-preserving change in risk. Thus, while a mean-preserving reduction in risk is a particularly simple characterization of technical change, it serves to illustrate that, under certain circumstances, a landlord will resist adopting a technical innovation. This arises because of the second best nature of the problem and the need to devise incentive schemes which balance risk-sharing with the preservation of work incentives.

\(^6\) The phenomenon of subsidized consumption credit was widely observed by Bardhan and Rudra (1978) in a sample of 110 villages in West Bengal.
V. Conclusion

This paper has argued that the interlinking of labor, output and credit contracts often observed in rural economies can be regarded as an attempt to improve allocative efficiency in the face of moral hazard. It was shown that all Pareto efficient allocations (regardless of their particular distributional characteristics) require a combination of wage-cum-output sharing with consumption credit contracts. The response of such contractual parameters to changes in underlying economic circumstances was also explored.

Further work could usefully focus (a) on extending these methods to treat the case of ex ante heterogeneous tenants and (b) on more dynamic situations where landlords and tenants have the opportunity of revising contracts in the light of reputations acquired in previous periods. 7/

Finally, there is the very interesting question of what policy conclusions are valid in a world where a rich panoply of rural institutions substitutes for the economist's landmark of a complete set of markets. We hope to address some of these issues in a subsequent paper.

7/ An interesting recent paper by Radner (1981) suggests that repeated games between principals and agents allow the former to surmount the monitoring problem under fairly stringent conditions.
Appendix 1

This appendix establishes that when agents with intertemporally additive utility functions are free to choose consumption and effort, \( \frac{de}{da} < 0 \) provided \( \phi_{ce} > 0 \). This fact is used in the proof of Proposition 8.

An agent's problem is to choose \( c \) and \( e \) to maximize

\[ W = \phi(c, e) + E[\alpha + \beta f(e, \theta) - c(1+i)] \]

This leads to

\[ W_c = \phi_c - (1+i)E\phi_c = 0 \]
\[ W_e = \phi_e + \beta E\phi_c e = 0 \]
\[ W_{cc} = \phi_{cc} + (1+i)^2 E\phi_{cc} < 0 \]
\[ W_{ee} = \phi_{ee} + \beta E[\phi_{cc}^2 e + \phi_{ce}^2 e] < 0 \]
\[ W_{ce} = \phi_{ce} - \beta(1+i)E\phi_{cc} e \]

To ascertain the effect of changing \( \alpha \) on consumption and effort, it is necessary to solve the following system:

\[
\begin{bmatrix}
W_{cc} & W_{ce} \\
W_{ec} & W_{ee}
\end{bmatrix}
\begin{bmatrix}
dc \\
de
\end{bmatrix}
= -
\begin{bmatrix}
W_{ca} \\
W_{ea}
\end{bmatrix}
\]

\[ W_{ca} = -(1+i)E\phi_{cc} > 0 \]
\[ W_{ea} = \beta E\phi_{cc} e < 0 \]
Appendix 1

From (A.1),

\[ \frac{de}{d\alpha} = \frac{1}{D} \begin{vmatrix} W_{cc} & -W_{ca} \\ W_{ec} & -W_{ea} \end{vmatrix} \]

where \( D \) is the determinant corresponding to the matrix on the left hand side of (A.1). From the second order conditions, \( D > 0 \). It is readily checked that \( \frac{de}{d\alpha} < 0 \) if \( \phi_{ce} > 0 \).
Appendix 2

This appendix briefly sketches the manipulations underlying the propositions established in section III of the text.

Agents' Problem:

An agent who sets \( e = 1 \) chooses \( c \) to maximize

\[
W = \phi(c) + p\phi[a+\delta Q-c(l+t)] + (1-p)\phi[a-c(l+t)]
\]  \hspace{1cm} (A.2)

This leads to:

\[
W_c = \phi' - (l+t)\{p\phi'_1 + (1-p)\phi'_0\} = 0
\]

\[
W_{cc} = \phi'' + (l+t)^2\{p\phi''_1 + (1-p)\phi''_0\} < 0
\]

\[
W_{ca} = -(l+t)\{p\phi''_1 + (1-p)\phi''_0\} > 0
\]

\[
W_{c\beta} = -(l+t)p\phi''_1Q > 0
\]

\[
W_{ct} = -i[p\phi'_1 + (1-p)\phi'_0] + ci(l+t)\{p\phi''_1 + (1-p)\phi''_0\} < 0
\]

\[
W_{ci} = -c[p\phi'_1 + (1-p)\phi'_0] + ci(l+t)\{p\phi''_1 + (1-p)\phi''_0\} < 0
\]

with \( d(pQ) = 0 \),

\[
W_{cp} = -(l+t)((\phi'_1 - \phi'_0) - \delta Q\phi''_1) > 0 \text{ if } \phi'_1 \text{ is convex because of the convex function inequality } (\phi'_1 - \phi'_0) \leq \delta Q\phi''_1.
\]

From the above,
Appendix 2, page 2

\[ \frac{dc}{d\alpha} = \frac{W_{c\alpha}}{W_{cc}} = \frac{(1+ti)[p\phi_1' + (1-p)\phi_0']}{W_{cc}} > 0 \]

\[ \frac{dc}{d\beta} = \frac{W_{c\beta}}{W_{cc}} = \frac{(1+ti)p\phi_1'Q}{W_{cc}} > 0 \]

\[ \frac{dc}{dc} = \frac{W_{c\bar{c}}}{W_{cc}} = \frac{i[(p\phi_1' + (1-p)\phi_0')] - c(1+ti)[p\phi_1' + (1-p)\phi_0']}{W_{cc}} < 0 \]

Note that

\[ \frac{dc}{dc} + ic\frac{dc}{d\alpha} = \frac{i[p\phi_1' + (1-p)\phi_0']}{W_{cc}} < 0 \]

\[ \frac{dc}{dt} = \frac{W_{c\bar{t}}}{W_{cc}} = \frac{c[(p\phi_1' + (1-p)\phi_0')] - c(1+ti)[p\phi_1' + (1-p)\phi_0']}{W_{cc}} < 0 \]

Substituting from above for \( W_{cc} \), using the first order conditions \( W_c = 0 \) and remembering that \( \phi = -\frac{c\phi''}{\phi} \), we may write

\[ ct + (1+ti) \frac{dc}{dt} > 0 \] according as \( \sigma \approx 1 \).

With \( d(pQ) = 0, \frac{dc}{dp} = \frac{W_{cP}}{W_{cc}} = \frac{(1+ti)(\phi_1' - \phi_0') - BQ\phi_1'}{W_{cc}} > 0 \)

Principal's Problem:

Choose \( \alpha, \beta, t \) to maximize

\[ F = \frac{1}{1+\bar{t}}[p(1-\beta)Q - \alpha - c(\alpha, \beta, t)(1-t)] \]

subject to
Appendix 2, page 3

\[ H = \phi[c + \beta Q - c(a, \beta, t)(1 + t_t)] - \phi[a - c(a, \beta, t)(1 + t_t)] - \frac{A}{p} \geq 0 \]

\[ R = \phi(c(a, \beta, t)) + p\phi[a + \beta Q - c(a, \beta, t)(1 + t_t)] + (1 - p)\phi[a - c(a, \beta, t)(1 + t_t)] - \bar{u} \geq 0 \]

where \( c(a, \beta, t) \) solves \((A.2)\).

The relevant derivatives for comparative statics are

\[ H_a = -\frac{(\phi'_1 - \phi'_0)\phi''}{W_{cc}} < 0 \]

\[ H_\beta = \phi'_Q - \frac{(1 + t_t)^2\phi'_1 p Q(\phi'_1 - \phi'_0)}{W_{cc}} > 0 \]

\[ H_c = i c(1 - \sigma) - 1 > 0 \quad \text{as } \sigma < 1 \]

\[ H_{\phi} = -i(\phi'_1 - \phi'_0)(c_t + (1 + t_t)\frac{dc}{dt}) > 0 \quad \text{as } \sigma > 1 \]

With \( d(pQ) = 0 \), \( H_p = \frac{1}{p}(\frac{A}{p} - \beta Q\phi'_1) - (1 + t_t)(\phi'_1 - \phi'_0)\frac{dc}{dp} \)

If \( H \geq 0 \) holds as an equality, i.e.,

\[ \frac{A}{p} = \phi'_1 - \phi'_0 \], it follows that

\[ \frac{A}{p} - \beta Q\phi'_1 = (\phi'_1 - \phi'_0 - \beta Q\phi'_1) > 0 \text{ because } \phi \text{ is concave and} \]

\[ H_p > 0 \]

\[ H_{\phi} = 0 \]

\[ R_a = p\phi'_1 + (1 - p)\phi'_0 > 0 \]

\[ R_\beta = p\phi'_1 Q > 0 \]
Appendix 2, page 4

\[ R_t = -icR_\alpha < 0 \]
\[ R_i = -ctR_\alpha < 0 \]
\[ R_p = (\phi - \phi_0) - \beta Q \phi' \geq 0, \text{ by concavity of } \phi. \]
\[ R_u = -1. \]

**Proposition 9 (proof):** With \( \beta \) constant, equation (33) of the text becomes

\[
\begin{bmatrix}
M_{\alpha} & M_{\alpha t} & H_{\alpha} & R_{\alpha} \\
M_{\alpha t} & M_{tt} & H_{t} & R_{t} \\
H_{\alpha} & H_{t} & 0 & 0 \\
R_{\alpha} & R_{t} & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
da \\
dt \\
d\eta \\
d\pi \\
\end{bmatrix}
=
\begin{bmatrix}
-M_{\alpha u} \\
-M_{tu} \\
0 \\
1 \\
\end{bmatrix}
\]

(A.3)

The determinant, \( D \), corresponding to the matrix on the left hand side of (A.3) is positive, by second order conditions. Consider the expression

\[ B_1 = H_{\alpha t} R_\alpha - H_{t} R_{\alpha} \]

\[ = - \frac{ic}{\sigma} H_{\alpha} R_\alpha > 0. \]

It follows from Cramer's rule that

\[ \frac{dr}{du} = \frac{1}{D} H_{\alpha} B_1 < 0 \]
Appendix 2, page 5

(ii) \( \frac{d\alpha}{du} = -H_B \beta > 0 \) according as \( \sigma > 1 \).

(iii) Since \( c = c(\alpha, \beta, t) \) and \( \beta \) is fixed,

\[
\frac{dc}{du} = \frac{dc}{d\alpha} \cdot \frac{d\alpha}{du} + \frac{dc}{d\beta} \cdot \frac{dt}{du} > 0 \text{ if } \sigma < 1.
\]

Proposition 10 (proof): With \( t \) constant, (33) becomes

\[
\begin{bmatrix}
M_\alpha & M_\alpha & H_\alpha & R_\alpha \\
M_\beta & M_\beta & H_\beta & R_\beta \\
H_\alpha & H_\beta & 0 & 0 \\
R_\alpha & R_\beta & 0 & 0
\end{bmatrix}
\begin{bmatrix}
da \\
d\beta \\
d\eta \\
d\pi
\end{bmatrix}
= 
\begin{bmatrix}
-M_\alpha u \\
-M_\beta u \\
0 \\
1
\end{bmatrix}
\frac{du}{du}
\]

The relevant determinant, \( D \), is again negative. Consider the expression

\[
B_2 = H_\alpha B_\beta - H_\beta B_\alpha < 0.
\]

From Cramer's rule,

(i) \( \frac{d\beta}{du} = \frac{1}{D} H_\alpha B_2 > 0 \)

(ii) \( \frac{d\alpha}{du} = \frac{1}{D} (-H_\beta) B_2 > 0 \)

(iii) \( \frac{dc}{du} = \frac{dc}{d\alpha} \cdot \frac{d\alpha}{du} + \frac{dc}{d\beta} \cdot \frac{d\beta}{du} > 0 \).

Proposition 11 (proof): With \( \alpha \) constant, (33) becomes
Appendix 2, page 6

\[
\begin{bmatrix}
M_{\beta \beta} & M_{\beta t} & H_{t} & R_{t} & d\beta \\
M_{t\beta} & M_{tt} & H_{t} & R_{t} & dt \\
H_{t} & H_{t} & 0 & 0 & d\eta \\
R_{t} & R_{t} & 0 & 0 & d\tau \\
\end{bmatrix}
= 
\begin{bmatrix}
-M_{\beta \gamma} \\
-M_{t \gamma} \\
0 \\
1 \\
\end{bmatrix}
data
\]

Notice that

\[B_{3} = H_{t} R_{t} - H_{t} R_{\beta} < 0 \text{ if } \sigma \geq 1.
\]

From Cramer's rule (using \( D \) to denote the relevant determinant, which is positive),

(i) \( \frac{d\beta}{du} = \frac{1}{D} (-H_{t})B_{3} \geq 0 \text{ if } \sigma \geq 1 \)

(ii) \( \frac{dt}{du} = \frac{1}{D} H_{t} B_{3} < 0 \text{ if } \sigma \geq 1 \)

(iii) \( \frac{d\epsilon}{du} = \frac{d\epsilon}{d\beta} \cdot \frac{d\beta}{du} + \frac{d\epsilon}{dt} \cdot \frac{dt}{du} > 0 \text{ if } \sigma \geq 1 \).

Proposition 12 (proof): With \( t \) constant, (34) becomes

\[
\begin{bmatrix}
M_{\alpha \alpha} & M_{\alpha \beta} & H_{\alpha} & R_{\alpha} & da \\
M_{\beta \alpha} & M_{\beta \beta} & H_{\beta} & R_{\beta} & d\beta \\
H_{\alpha} & H_{\beta} & 0 & 0 & d\eta \\
R_{\alpha} & R_{\beta} & 0 & 0 & d\tau \\
\end{bmatrix}
= 
\begin{bmatrix}
-M_{\alpha i} \\
-M_{\beta i} \\
-H_{i} \\
-R_{i} \\
\end{bmatrix}
data
\]

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From Cramer's rule

(i) \( \frac{dg}{d\lambda} = \frac{1}{d} (H_{l}R_{l} - H_{a}R_{l})B_{2} > 0 \) if \( \sigma < 1 \)

(ii) \( \frac{da}{d\lambda} = \frac{1}{d} (H_{b}R_{l} - H_{a}R_{l})B_{2} > 0 \) if \( \sigma \geq 1 \)

(iii) \( \frac{dc}{d\lambda} = \frac{dc}{da} \cdot \frac{da}{d\lambda} + \frac{dc}{d\sigma} \cdot \frac{d\sigma}{d\lambda} > 0 \) if \( \sigma = 1 \).

Proposition 13 (proof): With \( \beta \) constant, (36) becomes

\[
\begin{bmatrix}
M_{a} & M_{at} & H_{a} & R_{a} \\
M_{ta} & M_{tt} & H_{t} & R_{t} \\
H_{a} & H_{t} & 0 & 0 \\
R_{a} & R_{t} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
da \\
dt \\
dn \\
dr
\end{bmatrix}
= \begin{bmatrix}
-M_{ap} \\
-M_{tp} \\
-H_{p} \\
-R_{p}
\end{bmatrix}
\]

On remembering that \( d(pQ) = 0 \) and using Cramer's rule we have that

(i) \( \frac{dc}{dp} = \frac{1}{d} (H_{p}R_{a} - H_{a}R_{p})(H_{a}R_{c} - H_{c}R_{a}) \)

\[= -\frac{ic}{d\sigma} H_{R_{a}}(H_{R_{p}} - H_{R_{a}}) > 0 \] if the constraint (30) holds as an equality and \( \phi''' > 0 \).

(ii) \( \frac{da}{dp} = \frac{1}{d}(H_{t}R_{p} - H_{p}R_{t})(H_{a}R_{c} - H_{c}R_{a}) \)

\[= -\frac{ic}{d\sigma} H_{R_{a}}(H_{R_{p}} - H_{R_{a}}) > 0 \] if \( \sigma \geq 1 \), (30) is binding and \( \phi''' > 0 \).
References


