A LINEAR TWO-LEVEL PROGRAMMING PROBLEM

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1. INTRODUCTION

Multi-level programming, as defined in Section 2, can be viewed as a generalization of mathematical programming. This paper presents a brief review of the geometry of the multiple-level programming problem in Section 2. Algorithmic principles are discussed in Section 3. An algorithm is presented in Section 4, and is commented on in Section 5. A numerical example is given in Section 6, and concluding remarks follow in Section 7.

As a mathematical problem structure, multi-level programming is a general problem. It happens to correspond to a primordial economic policy problem since:

1. Higher level decision makers have direct control of some variables (the policy variables), but not all variables.

* With the usual caveat, the constructive comments of J. Falk and Stephen Robinson are fully acknowledged.

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Thus, policy makers may be able to control directly tax rates, and the size of the government’s budgetary deficit.

ii) Lower level decision makers have direct control of other variables (the behavioral variables), which are manipulated in the light of the levels of the policy variables. Thus, behavioral variables such as rate of private investment, and agricultural production may be decided by many decentralized decision makers, following their own behavioral rules.

iii) The higher level decision (policy) makers wish to influence a third set of "impact" variables which are frequently not under the direct control of either decision making group. Thus, while unable to directly control their levels, policy makers may be interested in influencing the level of impact variables such as pollution, unemployment, rate of inflation and balance of payments.

This problem structure is in marked contrast to mathematical programming, where all variables are under direct control of one decision maker. Multi-level programming, as defined in this paper, refers to a problem situation wherein 'outer' decision makers affect the solution space of inner decision makers within a strictly hierarchical structure. The definition of multi-level programming could be widened to include the problem situation wherein the outer decision makers also affect the parameters of the inner objective functions, e.g., by product or input price; or, indeed, factor-product transformation rates, via research or extension. In this paper we prefer to restrict our discussion to the 'narrow' definition of multi-level programming; we propose to discuss the 'wider' problem in a subsequent paper.
Illustrative results using the problem structure described in this paper have been presented by Candler and Norton [2] for a 50-row by 300-column model of agricultural production in a region of Mexico. This example showed that the technological frontier, of what could be achieved if policy makers controlled all variables, can easily lie twice as far from the origin, as the policy frontier, of what can be achieved with only the variables currently under the control of the policy maker.

Enough has been said to indicate that multi-level programming is not only an interesting mathematical problem, but also a problem of major significance for economic policy makers. Related problem structures have also been investigated in connection with armed conflict [1]. The problem structure can be tied both to game theory and mathematical programming.

The two-stage problem can be viewed as a two-person, nonzero sum, two-move game, with the first mover's (policy player's) decisions affecting the second mover's (behavioral player objective function value, and his feasible region. The second mover's optimal decision then affects the value of the first mover's objective function. Thus the policy maker must account for subsequent optimal responses by a second optimizer over which the policy maker has only partial control.

Viewed as a mathematical program from the policy maker's standpoint, the problem treated herein is that of maximizing a (generally non-concave) piecewise linear function over a polyhedron. The example treated in Section 6 has two isolated local solutions.

1/ Though not the authors' area of expertise, many ecological problems would appear to have this problem structure.
The problem addressed here is related to the linear max-min problem studied by Falk[4]. Indeed, this latter problem is a special case of the two-level problem of this paper in which the behavioral objective function is in direct opposition to the policy objective function. As a result, Falk's problem is equivalent to maximizing a piecewise linear convex function over a polyhedron, while the implied objective function of this paper need not be convex.

The problem considered can have local optima, hence there is little hope of a monotonic improvement in the objective function. Rather some form of implicit enumeration has to be relied on, using the global (but necessary condition) information generated in the search for locally better solutions.

By paying attention to all necessary condition information generated to date, we can avoid returning to any previously explored basis. This provides the rationale for the proposed algorithm which is, in essence, simply an implicit search.

This paper assumes the absence of degeneracy. This is an important assumption, as can be seen from the following problem:

maximize \((-x + y_1 - y_2)\)

subject to \(0 \leq x \leq 1\)

where \(y_1, y_2\) solve

maximize \((y_1 + y_2)\)

subject to \(y_1 + y_2 = 1 - x\)

\(y_1, y_2 \geq 0\).
In this cast, the solution to the inner problem, in \( y_1 \) and \( y_2 \), is not unique: any values of \( y_1 \) and \( y_2 \) which satisfy the restraints yield the same value of \( y_1 + y_2 \). For the outer problem, in \( x \), the choice of \( y_1 \) or \( y_2 \) is crucial. This type of degeneracy is not treated in this paper. Its proper solution would require the possibility of influencing the objective function of the inner problem (to restore uniqueness), or some knowledge of the probabilities of alternative solutions being chosen.

The linear max-min problem studied by Falk[4], avoids this problem, since if the inner decision maker is indifferent between solutions, so is the outer decision maker.

The problem discussed below can be expressed in a number of different ways. Some of our colleagues prefer to describe it as a problem in "vested optimization", whilst others think of it as "parametric perturbation of the right hand side". Stephen Robinson described the problem as:

"This paper deals with the problem

\[
\max \left\{ f(y(x), x) \mid x \geq 0 \right\}
\] (1)

where

\[
y(x) := \text{avg} \max \left\{ g(y) \left| H_1 y = b - H_2 x, y \geq 0 \right\}
\] (2)

Here \( f \) and \( g \) are real valued functions, and \( f \) is defined only for values of the second argument which lie in the domain of \( y \) (it could be considered to be \( \infty \) for all other \( x \) ...) The problem is difficult, since ... in general it does not exhibit the nice convexity properties
that one would usually associate with linear programming; of course, as can be seen from (1), it is not a linear programming problem at all, but rather a nonconvex, non-linear one).

Some readers will find this description, which is correct, more helpful than our problems $P_1$ and $P_2$, below. For readers who find (1) and (2) somewhat opaque, we suggest continuing to problem $P_2$ and, indeed, the numerical example in Section 6.

The general multi-level programming problem, as it affects the restraints of the problem, is stated in Section 2.1, together with the two-level problem this paper addresses. These problems are not L.P. problems. Section 2.2 defines three associated L.P. problems. Since these three structures, particularly $P_3$ and $P_4$, are crucial to our algorithmic proposals, it may be helpful to note this section for review, as required. Similarly, Section 2.3 defines a BOB, "behaviorally optimal basis", and FBOB, "feasible behaviorally optimal basis". These definitions are again crucial to the latter discussion. Behavioral optimality could, perhaps, be better expressed as "behaviorally dual feasible", since it carries no implication of (primal) feasibility - however, we have stayed with our acronyms since replacing FBOB by the more correct "primal feasible, behaviorally dual feasible, basis" would, with frequent repetition, become cumbersome. Also, whilst for a L.P. problem, in the absence of degeneracy, there is only one primal 2nd dual feasible basis, the possibility of perturbing the right hand side, means that in the two-level problem, there are many, a small infinity, of such bases.
2. GEOMETRY OF THE LINEAR TWO-LEVEL PROGRAMMING PROBLEM

2.1 Problem Definition

For the k-level programming problem we have a (1 x n) solution vector $x$ partitioned into $k$ sub-vectors:

$$x = (x_1, x_2, \ldots, x_k)$$

Define

$$x_j = (x_{j+1}, x_{j+2}, \ldots, x_k)$$

We can now state the k-level programming problem:

**P1** - Find $x$ where

$x_k$ solves: maximize $g_k(x)$

subject to $h_i(x) = 0 \quad i = 1, \ldots, m$ \hspace{1cm} (1.2)

$x \geq 0 \quad \hspace{1cm} (1.3)$

and where $x_j$ solve $\max \left[ g_j(x) \mid x_j \right] \quad j = k - 1, \ldots, 1$ \hspace{1cm} (1.4)
If \( k = 1 \), this reduces to the normal statement of the mathematical programming problem.

If \( k = 2 \) and all functions are linear, the case for which an algorithm is offered, the problem can be written:

\[ P_2 \] Find \((x_1, x_2)\) such that:

\[ x_2 \text{ solves: maximize } c_2 x \]

subject to \( H_1 x_1 + H_2 x_2 = b \)

\[ x \geq 0 \]

and where \( x_1 \) solves \( \max \left[ x_1 \right] \)

\[ \text{where } H_1 \text{ is } (m \times n_1) \text{ and } H_2 \text{ is } (m \times n_2). \]

Thus, in problem \( P_2 \), \( x_2 \) must be set so that:

1) \( H_1 x_1 + H_2 x_2 = b \)

\[ x_1, x_2 \geq 0 \]

is feasible for some \( x_1 \), and, for all such \( x_2 \),
ii) $x_2$ yields the highest value of $c_2x$, with the understanding that $x_2$ will be set optimally in a different and subsequent optimization.

As noted earlier, this paper assumes that for a given $x_2$, there will be a unique solution $x_1$.

We will denote a typical column vector in $H_1$ by $h_1$, and a typical column vector in $H_2$ by $h_2$. For convenience we assume rank $(H_1) = m$.

Thus we have:

$$x = (x_{11}, \ldots, x_{1j}, \ldots, x_{1n_1}, x_{21}, \ldots, x_{2q}, \ldots, x_{2n_2})$$

= (behavioral variables, policy variables)

A similar nonconvex programming problem, (formulated as a max-min problem with a single objective function $f(x_1, x_2)$), has been described by Falk[4]. The algorithm described here can be used to solve the Falk's problem, and a trivial generalization of his algorithm would allow solution of the general linear two-level problem.
2.2 Related L.P. Problems

We now present three (primal) L.P. problems that are associated with the two-level programming problem, P2.

P3 - For any given \((2^{th})\) set of non-negative values for the policy variables, \(x_2 = \frac{x^{(I)}}{x_2} > 0\), find values for the behavioral variables, \(x_1\), such that:

\[
f_1 = \max_{x_1} \left[ c_1 (x_1, \frac{x^{(I)}}{x_2}) \right]
\]

subject to

\[
H_1 x_1 = b - H_2 \frac{x^{(I)}}{x_2}
\]

\(x_1 \geq 0\)
Of course there is no guarantee that \( P_3 \) will have a feasible solution for any given set of values: \( x_2^{(\lambda)} \geq 0 \). We refer to \( P_3 \) as the "behavioral" or "inner" L.P. problem. 

\[ \text{P4 - Given a basis set } B_\lambda \text{ from } H_1, \text{ that is optimal with respect to the behavioral L.P. problem: find values of } (x_1^{(\lambda)}, x_2) \text{ such that:} \]

\[
f_2 = \max_{x_1^{(\lambda)}, x_2} \left[ c_2 (x_1^{(\lambda)}, x_2) \right]
\]

subject to

\[
B_\lambda x_1^{(\lambda)} + H_2 x_2 = b
\]

\[
x_1^{(\lambda)}, x_2 > 0
\]

Note that this "policy" or "outer" L.P. problem is defined only for behavioral variables (activities) that are members of \( B_\lambda \).

\[ \text{P5 - Find values of } x_1 \text{ and } x_2 \text{ such that:} \]

\[
f_2 = \max_{x_1, x_2} c_2 x
\]

subject to

\[
H_1 x_1 + H_2 x_2 = b
\]

\[
x_2 > 0.
\]

Clearly the solution to \( P_5 \) places an upper bound on the solution to \( P_2 \).
In Falk's problem the optimal solution has to be both a vertex of the constraints of \( P_5 \), as well as a vertex of the projection of this set onto the policy space. In the present paper, an optimal setting for the policy variables could be interior to this projection.

2.3 Two Definitions

BOB: Any basis set \( B_\mathcal{L} \), from \( H_1 \), that satisfies

\[ c_{1\ell} B_\mathcal{L}^{-1} H_{1j} - c_{1j} 0 \quad \text{for all} \quad j = 1, \ldots, n_1 \]

is a behaviorally optimal basis, or BOB.

In particular if \( P_3 \) has a behaviorally optimal solution, which may or may not be feasible, the associated basis \( B_\mathcal{L} \) is BOB.

Note that behavioral optimality is unaffected by the settings for the policy variables \( x_2 \). The values of the policy variables only affect the feasibility of a behavioral basis.

FBOB: Any BOB basis \( B_\mathcal{L} \), for which \( P_4 \) has a feasible solution, is a feasible behaviorally optimal basis, or FBOB.

Eases, from \( H_1 \), which give a feasible solution to \( P_3 \), or \( P_4 \), but which are not BOB, are of no interest. Since such bases would never be chosen by the behavioral decision makers, regardless of the setting of \( x_2 \).

2.4 Feasible Solutions to \( P_2 \)

If problem \( P_3 \) has an optimal feasible solution \( x_{1\mathcal{L}}(\mathcal{L}) \), then \( x = (x_1^{(1)}, x_2^{(\mathcal{L})}) \) is a feasible solution to the two-level linear programming problem, \( P_2 \).
If there is no feasible solution to \( P3 \), then \( x_2 = \tilde{x}_2^{(\ell)} \) is 
not a feasible setting for the policy variables in \( P_2 \).

Thus, a feasible solution to \( P2 \) is defined as any \( x_2 = \tilde{x}_2^{(\ell)} \)
satisfying (2.2) and (2.3), and where problem \( P3 \) has an optimal feasible solution for \( x_2 = \tilde{x}_2^{(\ell)} \geq 0 \), and \( x_2^{(\ell)} \) is the solution.

If \( x_1 = \tilde{x}_1^{(\ell)} \) is a feasible, but non-optimal solution to \( P3 \) given \( x_2 = \tilde{x}_2^{(\ell)} \), then \( x = (\tilde{x}_1^{(\ell)}, \tilde{x}_2^{(\ell)}) \) is not a feasible solution to \( P2 \).

The (feasible) solution space for \( P2 \), is then \( \{ x_2 \geq 0 \mid b_{1}^{-1} b - B_{1}^{-1} H_2 x_2 \geq 0, \ c_{1}^{-1} B_{1}^{-1} H_1 j - c_{1} j \geq 0 \ \text{for all } j = 1, \ldots, n_1; \ B \text{ a basis from } H_1 \} \). This is a convex polyhedron.

Now, if \( B_2 \) is a behavioral optimum basis (BOB), then from (3.2), \( B_2 \) will be a feasible behavioral optimum basis (FBOB) if there exist values of the policy variables such that:

\[
B_{1}^{-1} H_2 x_2 \leq B_{1}^{-1} b 
\]  
(6)

since, \( x_2^{(\ell)} = B_{1}^{-1} b - B_{1}^{-1} H_2 x_2 \).

Thus for values of \( x_2 \geq 0 \) satisfying (6), the corresponding solution to the behavioral L.P. problem, \( P3 \), is given by (7).

Therefore, given (BOB): \( B_2 \), values for \( (\tilde{x}_1^{(\ell)}, \tilde{x}_2^{(\ell)}) \) satisfying the constraint set for the policy L.P. problem \( P4 \), i.e., satisfying (4.2) and (4.3), are feasible solutions to \( P2 2/ \).

2/ If the intersection of the sets of values for \( x_2 \) given by (6) and the non-negativity condition, is empty, then there is no set of values for the policy variables such that this particular (BOB): \( B_2 \) gives a feasible solution to the behavioral L.P. problem, \( P3 \), and hence contributes to a feasible solution to the two-level programming problem, \( P2 \).
2.5 Optimal Feasible Solutions to P2

Let \( x^*_2 = \ldots \) be an optimal feasible setting for the policy variables in the two-level programming problem, \( \mathbf{P}_2 \). Then, from the definition of \( \mathbf{P}_2 \), there exists a basic optimal feasible solution to the behavioral L.P. problem \( \mathbf{P}_3 \):

\[
x^*_1 = B^*_1 \quad b - B^*_1 H_2 x^*_2 > 0 \quad \text{for} \quad (x^*_1 | H_1 \in (\text{BOB}) \quad B^*_1) \quad (8)
\]

\[
= 0 \quad \text{otherwise},
\]

such that \( f^*_2 = c_2(x^*_1, x^*_2) \) is the optimal value for the policy (outer) objective function (2.1).

Consider now the policy L.P. problem \( \mathbf{P}_4 \) associated with \( \mathbf{P}_4 \), and let \( \mathbf{W}_* \) denote an optimal feasible basis for \( \mathbf{P}_4 \), and let \( f^*_2 \) denote the corresponding value of the policy objective function. Then \( f^*_2 = f^*_2 \), since if \( f^*_2 > f^*_2 \), \( f^*_2 \) is not the optimal value for the policy objective function for the two-level programming problem \( \mathbf{P}_2 \). This result proves Theorem 1.

3/ As noted earlier, in order to avoid complexities associated with indeterminate solutions to the multi-level programming problem \( \mathbf{P}_2 \), we assume that the optimal solution to the behavioral L.P. problem, \( \mathbf{P}_3 \), \( x^*_2 = x^*_2 \), is unique. Strictly it is only necessary to assume that the value of \( f^*_2 \), for any optimal feasible solution to \( \mathbf{P}_3 \), with \( x^*_2 = x^*_2 \), is unique.

4/ Clearly \( f^*_2 \geq f^*_2 \) since \( (x^*_1, x^*_2) \) is a feasible solution to \( \mathbf{P}_4 \), given (BOB): \( \mathbf{B}_* \).
Theorem 1

Given there exists an optimal feasible solution to the two-level programming problem P2, there exists a (BOB): \( B_\star \) such that the corresponding basic optimal feasible solution to the policy L.P. problem P4 is an optimal feasible solution to P2.

As a corollary to Theorem 1 we have:

Corollary

An optimal feasible solution to the linear two-level programming problem P2 can be represented as a basic feasible solution to problem P5.

This result follows since \( W_\star \), defined above, is a feasible basis for problem P5. Therefore, as Falk [4] has already noted, an examination of all feasible basic solutions to problem P5 constitutes an algorithm that will find the solution to P2 in a finite number of steps. We can add that examination of all behavioral optimal bases will also find the solution to P2 in a finite number of steps.

Whereas the branch and bound algorithm presented by Falk, for solving a subset of our two-level programming problem P2, concentrates on the relationships between P3, P5 and P2, the algorithm presented here concentrates on the relationship between P4 and P2.

Also, if \( W_\star \) is an optimal feasible basis for the policy L.P. problem P4 corresponding to (BOB): \( B_\star \), then:

\[
(x_1^*, x_2^*)_p = \begin{cases} 
W_\star^{-1} \gamma & \text{for } (x_1^*, x_2^*) \mid H_{1j}, \ H_{2q} \in W_\star \text{, } (9) \\
0 & \text{otherwise},
\end{cases}
\]
is an optimal feasible solution to the two-level programming problem $P_2$.

If the optimal setting for the policy variables is unique, then:

$$x_{2q}^* = x_{2q}^{**} \quad \text{for} \quad H_{2q} \in W_*$$

$$= 0 \quad \text{otherwise},$$

and,

$$x_{1j}^* = x_{1j}^{**} \quad \text{for} \quad H_{1j} \in B_* \cup W_*$$

$$= 0 \quad \text{otherwise}.$$ 

(See Appendix I for a more complete discussion of (10) and (11).)

At this point some readers may prefer to skip to the proposed algorithm in Section 4, so as to motivate the ensuing geometric, or algebraic, sections.

2.6 Geometry of a High-Point in (BOB): $B_2$

Let $B_2$ be a (BOB) and assume that

$$S_2: \{x_2 \mid B_2^{-1} H_2 x_2 \leq B_2^{-1} b ; x_2 \geq 0\}$$

is not empty, i.e., corresponding to $B_2$ there is at least one set of values for $x_2$ (the policy variables) such that there exists a feasible solution to the multi-level programming problem, $P_2$. 
Consider now the policy L.P. problem, P4, corresponding to \( B^*_2 \). Since we have assumed that there is a feasible solution to \( P4 \), then either \( P4 \) is unbounded or it has an optimum basic feasible solution. If \( P4 \) is unbounded, then \( P2 \) is also unbounded. If \( P4 \) has an optimal basic feasible solution, we refer to it as the high-point in basis \( B^*_2 \), in the sense that it yields the highest value of \( f^*_2 \) (the outer objective function for our two-level programming problem \( P2 \)), that can be achieved without a change in the behavioral basis.

As discussed earlier, a feasible solution to the policy L.P. problem \( P4 \) associated with (BOB): \( B^*_2 \), is a feasible solution to the two-level programming problem \( P2 \). We have also shown (Theorem 1) that an optimal solution to the two-level programming problem must also be the optimal solution to a policy L.P. problem, i.e., there is some (BOB): \( B^*_x \) such that the solution to the policy L.P. problem corresponding to \( B^*_x \), is an optimal feasible solution to the two-level programming problem \( P2 \).

Let \( W^*_x \) be an optimal feasible basis for the policy L.P. problem \( P4 \); i.e., \( W^*_x \) is an optimal feasible basis set from \([B^*_x, H^*_2] \) where we assume \( S^*_x \) is not empty. The corresponding (high-point) solution values for \( P4 \) are:

\[
(x^*_1, x^*_2) = W^{-1}_x b \quad \text{for} \quad H^*_1, H^*_2 \in W^*_x
\]

\[
= 0 \quad \text{otherwise.}
\]

\[5/ \text{ We have assumed } S^*_x \neq \emptyset \text{ for (BOB): } B^*_x.\]
Denote the corresponding high-point value for the policy objective function by \( f^*(l) \).

The set of reduced cost values for the policy objective function, (the objective function for \( P_4 \) and \( P_2 \)), associated with \( W_l \) may be classified as follows:

\[
\begin{align*}
    f_{2q}^{(l)} & = 0 & H_{2q} & \in W_l & (13.1) \\
    f_{2q}^{(l)} & > 0 & H_{2q} & \notin W_l & (13.2) \\
    f_{2j}^{(l)} & = 0 & H_{1j} & \in B_l, W_l & (13.3) \\
    f_{2j}^{(l)} & > 0 & H_{1j} & \notin B_l, \notin W_l & (13.4) \\
    f_{2j}^{(l)} & < 0 & H_{1j} & \in B_l & (13.5)
\end{align*}
\]

Conditions (13.1) to (13.4) hold since \( W_l \) was assumed to be an optimal (feasible) basis for the policy L.P. problem \( P_4 \) corresponding to (BOB): \( B_l \). Condition (13.5) holds since the behavioral vectors (activities) that did not belong to \( B_l \), were not considered as eligible activities for the policy L.P. corresponding to (BOB): \( B_l \).

If in (13.5) \( f_{2j}^{(l)} \geq 0 \) for all \( H_{1j} \notin B_l \), then clearly \( W_l \) is an optimal feasible basis for the L.P. problem \( P_5 \). But the solution to the L.P. problem \( P_5 \) is an upper bound on \( f_2 \) in \( P_2 \), and hence, under this condition, \( W_l \) is a global optimum feasible basis for the multi-level programming problem, \( P_2 \).

Unfortunately, \( f_{2j} \geq Q \) for (13.5) is a sufficient, but not a necessary condition for a solution to the multi-level programming problem, \( P_2 \).

---

6/ Remembering that the subscripts \( j \) and \( q \) denote behavioral and policy variables, respectively.
This sufficient condition does yield a necessary condition for a better solution. The basis for any higher value of \( f_2 \) must contain at least one vector such that \( f_{2j}^{(k)} < 0 \) in (13.5). We now define a set \( T_{k}^{1} \) corresponding to the high-point solution in (FBOB): \( B_{k} \).

We now define a set \( T_{k}^{1} \) corresponding to the high-point solution in (FBOB): \( B_{k} \).

\[
T_{k}^{1} = \{ H_{lj} \mid f_{2j}^{(k)} < 0 \} \tag{14}
\]

Thus, any behavioral optimum basis (OB) leading to a better high-point value for \( f_{2} > f_{2}^{*} \), must contain at least one vector from \( T_{k}^{1} \). Unfortunately, this is a necessary and not a sufficient condition for obtaining a better high-point value of \( f_2 \). This is, however, a necessary condition which applies globally:

**Theorem 2**

If \( W_{k} \) and \( W_{l} \) are optimal feasible bases for problem \( P_{4} \) corresponding to (OB): \( B_{k} \) and \( B_{l} \) respectively, and \( f_{2}^{*(k)} > f_{2}^{*(l)} \), then \( B_{k} \) contains at least one \( H_{lj} \in T_{k}^{1} \).

**Proof**

\[
f_{2}^{*} = f_{2}^{*} - \sum_{j=1}^{m} x_{j}^{l} + \sum_{q=1}^{m} x_{q}^{l} \tag{15}
\]
Now, from equations (13) we have:

\[
\begin{align*}
q_{2q} & \geq 0 \\
q_{2j} & \geq 0
\end{align*}
\]

Thus, if \( f_2 > f^*_2 \), then \( x_{1j} > 0 \), \( f_{2j} < 0 \) for at least one \( H_{1j} \notin B_k \). But if \( f_2 > f^*_2 \), this is exactly the requirement that \( W_k \), and hence \( B_k \), contain at least one \( H_{1j} \in T^1 \).

It should be emphasized that, due to the non-convex problem structure of \( P_2 \), \( T^1_\ell \) only provides a necessary condition for improvement in the value of \( f_2 \). For a given linear two-level programming problem there may be many (FBOBs) which contain at least one vector from \( T^1_\ell \) and have high-point values with \( f_2 \leq f^*_2 \). There will not, however, be any (FBOBs) with high-point values with \( f_2 > f^*_2 \) which do not include at least one vector from \( T^1_\ell \).

Also, given a set of (FBOBs): \( B_1, \ldots, B_\ell \), and their corresponding \( T^1 \) sets: \( T^1_1, \ldots, T^1_\ell \); then if (FBOB): \( B_{\ell+1} \) is to have a higher high-point value for \( f_2 \) than any previous (FBOB), then it is necessary that \( B_{\ell+1} \) contain an element of \( T^1_1 \), and an element of \( T^1_2 \), and \( \ldots \), and an element of \( T^1_\ell \).

\[7/\] Of course, if a particular behavior21 activity, say \( H_{1r} \), was a member of each of the \( T^1 \) sets: \( T^1_1, \ldots, T^1_\ell \), then inclusion of this one behavioral activity in \( B_{\ell+1} \) would satisfy the necessary \( T^1 \) set conditions.
We may note here that the $T^1_k$ set necessary condition can be represented by a constraint:

$$\sum_{j=1}^{n_1} \delta_{kj} y_j \geq 1$$  \hspace{1cm} (17)

where $\delta_{kj} = 1$ if $H_{1j} \in T^1_k$

= 0 otherwise,

and the $y_j's$ are (0, 1) variables.

The existing $T^1$ sets at any stage, can be represented by a set of constraints for an integer L.P. problem. The problem of satisfying the $T^1$ sets is the same as finding a feasible solution to this integer L.P. problem (see Appendix III).

If there is no (FBOB): $B_{k+1}$ satisfying the necessary $T^1$ set constraints for a better (FBOB) than any previous (FBOB), then the optimal solution to the linear two-level programming problem $P2$ is given by the previous best high-point solution.

Let $[T^1_k]$ represent the set of $k$ constraints corresponding to $T^1_1$, ..., $T^1_k$. 
In the next section we examine the opportunity for making some monotonic improvement in the value of the objective function for the linear two-level programming problem, using the information contained in only the most recent \( T^1 \) set. Subsequent sections examine the problem of constructing a (FBOB) subject to the constraint of satisfying all previous \( T \) sets, i.e., satisfying \( [T^1]_k \).

2.7 Geometry of Neighboring Bases

A high point in basis, \( B_k \), is defined as the optimal solution to problem \( P4 \). It is possible that this occurs "at the origin in policy space", \( x_2^{(l)} = 0 \). In this case, provided (13.2) and (13.4) hold as strong inequalities (i.e., in the absence of degeneracy in the policy objective function), the high-point will be a local optimum. In the presence of degeneracy, at least one policy variable can be introduced into the basis, with the implications discussed below.

In general \( x_{2q} > 0 \) for at least one policy variable at a high-point, and correspondingly, an equal number of \( x_{lj} = 0, \ H_{lj} \in B_k \). Thus at a high-point \( x_2^{(l)} \), there will in general be several zero elements on the RHS of the behavioral problem \( P3 \) and there will be the opportunity for alternative bases to be (FBOB) at the high-point. Any basis to the behavioral problem \( P3 \) which can be obtained by swapping activities at zero level at \( x_2^{(l)} \), will be called a (primal) feasible basis at \( x_2^{(l)} \). Any feasible basis at \( x \), which is also-optimal will be called an adjacent basis at \( x_2^{(l)} \). All adjacent bases are thus FBOB. Any adjacent basis at \( x_2^{(l)} \) which can be obtained by a single pivot operation will be called a neighboring basis for \( B_k \).

\( \text{See Appendix I.} \)
If \( x_2^{(\ell)} \) is the high-point for all adjacent bases at \( x_2^{(\ell)} \), then \( x_2^{(\ell)} \) corresponds to a local optimum of \( P_2 \), since any small change in \( x_2 \) away from \( x_2^{(\ell)} \) will lead to lower values of \( f_2 \).

In order to reach a feasible basis to \( B_2 \) at \( x_2^{(\ell)} \), it is only necessary to replace a vector \( h_{1j} \in B_2 \), by another vector \( h_{1r} \notin B_2 \), where \( h_{1j} \) is the basis at zero level. Provided \( h_{1r} \) is not a linear combination of the \( m-1 \) vectors remaining in the basis, this substitution will give an alternative basis \( B_{2r} \). Since \( x_2^{(\ell)} \) is a high-point, or point of "induced degeneracy" from the viewpoint of the behavioral problem, and given that \( h_{1j} \) was in the basis at zero level, \( h_{1r} \) will equally be in the basis at zero level, and the alternative basis will be feasible. As defined above, a neighboring basis is, however, not only feasible, but also (FBOB).

Consider now the solution to the behavioral L.P. problem, \( P_2 \), corresponding to \( B_2 \), and given \( x_2 = x_2^{(\ell)} \). Define:

\[
R_2 = \left\{ i \mid x_{1i}^{(II)} = C, \ h_{1i} \in B_2 \right\}
\]

where, \( x_{1i}^{(II)} = B_{2i}^{-1} b - B_{2i}^{-1} H_2 x_2^{(II)} \). Thus \( R_2 \) identifies the degenerate rows (vectors) of the updated tableau of the behavioral L.P. problem \( P_3 \), corresponding to (FBOB): \( B_2 \), at the high-point setting for the policy variables \( y^{(\ell)} \). As indicated above, (and in Appendix I), a sufficient condition for \( R_2 \neq \emptyset \) is that \( W_2 \) contain at least one vector from \( H_2 \).

If \( h_{1i}, i \in R_2 \), can be replaced by \( h_{1r} \notin R_2 \) to give a neighboring basis \( B_{2r} \), and if \( h_{1r} \) is a member of \( T_{1r} \) (defined in (14)), then \( B_{2r} \) will have a high-point such that \( f_2^{*}(\ell r) > f_2^{*}(\ell) \). For
convenience we refer to $B_{kr}$ satisfying these conditions as a better neighboring basis to $B_k$.

Let:

$$d_{ij} = \left( \frac{t_{ij}}{t_{ij}} \right) \quad \text{if } t_{ij} < 0, \quad \text{i.e., } R_j, \quad H_{lj} \notin B_k$$

where $t_{ij}$ is the $i$th element of $-B_k^{-1}H_{lj}$, and $r_{ij} > 0$ is the corresponding reduced cost value for the behavioral objective function. Then $H_{lr}$ can replace $H_{li}$ in $B_k$ to yield a neighboring basis $B_{kr}$, where $H_{lr}$ is identified by:

$$d_{lr} = \min_j d_{ij} \quad (20)$$

If $H_{lr} \in T_k^1$, then $B_{kr}$ is a better neighboring basis to $B_k$. Also, for given $i$ and in the absence of dual degeneracy in the behavioral L.P. problem, $H_{lr}$ identified by (20) will be unique. Clearly, if $t_{ij} > 0$ for all $j$, $H_{li}$ cannot be replaced in $B_k$ to yield a neighboring basis.

The vectors in $B_k$ corresponding to $i \in R_k$ can be examined in turn to derive the set of neighboring bases to $B_k$. If none of the neighboring bases to $B_k$ contain a member of $T_k^1$, then $B_k$ yields the high-point for all neighboring bases. However, this high-point is not necessarily a local optimum since one or more adjacent bases (requiring the substitution of two or more vectors in $B_k$) may have a higher high-point.

As noted earlier, if there is dual degeneracy, i.e., $H_{lj} = 0$ for at least one $H_{lj} \notin B_k$, then the response of the inner decision maker to given policy setting $x_2$ is indeterminate (since the inner decision maker is indifferent as between degenerate choices).
Our interest in identifying better neighboring bases to $B_\ell$ is generated by the possibility of making at least some monotonic improvement in the value of the objective function for $P_2$.

2.8 Geometry of T-Sets

We have just seen that $T_\ell^1$, associated with (FBOB): $B_\ell$, enables us to identify a better neighboring (FBOR) to $B_\ell$, should one exist. In this section we discuss the problem of finding (FBOB): $B_\ell$ satisfying all previous $T_\ell^1$ set constraints. Our procedure is to find a set of vectors from $H_1$ satisfying all previous $T$-sets, and to then find a (BOB) that includes this set. We then check this (BOB) for feasibility.

Let $C_\ell$ be a $(m \times m)$ matrix of vectors from $H_1$ satisfying $[T_\ell^1]_{k}^{10/}$. Since there exists some (BOB): $B_\ast$, such that the high-point in $B_\ast$ is a solution to $P_2$ we can apply the constraint:

$$\sum_{j=1}^{n_1} y_j \leq m$$

(21)

where $y_j$ is integer, $(0, 1)$, to the selection of any $C$ matrix.\(^{11/}\)

Thus, $m_1 \leq m$.

\(^{10/}\) We may note that the $H_1^{10/}$, the optimal feasible basis for $P_5$, will automatically satisfy $[T_\ell^1]_{k}^{10/}$, since this solution provides an upper bound on $P_2$.

\(^{11/}\) Of the $n_1$ vectors in $H_1$, at most $m$ can be selected to satisfy existing $T$-set constraints.
Our next task is to check whether or not we can find a \((\text{BOB})\):

\[ l = [C_\lambda, D_\lambda], \]

where \( D_\lambda \) is a set of \((m - m_\lambda)\) vectors from \( H_\lambda \). We describe two ways of performing this check.

For the behavioral L.P. problem (primal), \( P_\lambda \), attempt to obtain a basis containing the vectors in \( C_\lambda \), (using non-zero pivots in the simplex method). If we cannot obtain a basis for \( P_\lambda \) containing \( C_\lambda \),

then we can add the constraint:

\[
\begin{align*}
   \Gamma^2 & : \sum_{j \in C_\lambda} y_j \leq m_\lambda - 1
\end{align*}
\]

where \( y_j \)'s are \((0, 1)\) variables, to our selection of \( C_{\lambda+1} \).

If we obtain a basis for \( P_\lambda \) containing \( C_\lambda \), e.g.

\[
   B_\lambda = [C_\lambda, D_\lambda],
\]

then either \( B_\lambda \) is \((\text{BOB})\) or at least one \( r(\lambda) \) \( l_j \) \( \leq 0 \). Assume \( r(\lambda) \) \( l_j \) \( < 0 \), and define \( t \) as the \( i \)th element in \( B_\lambda^{-1} H_{lr} \). For convenience we assume that the first \( m_\lambda \) vectors of \( B_\lambda \) are \( C_\lambda \). Now, if \( t \) \( l_i \) \( r \) \( > 0 \) for at least one \( i > m_\lambda \), we introduce \( H_{lr} \) into the basis, and this process can be repeated until either a \((\text{BOB})\) containing \( C_\lambda \) is found, or the introduction of any \( \text{vector} \) with a \( \text{negative} \) \( \text{reduced cost value} \) \text{would replace} a member of \( C_\lambda \).

In this latter case we can again apply the constraint \((22)\) to our selection of \( C_{\lambda+1} \).

Alternatively, we can state the problem of finding \((\text{BOB})\):

\[
   B_\lambda = [C_\lambda, D_\lambda]
\]

in terms of finding a basic feasible solution to the dual of the behavioral L.P. problem, subject to additional constraints on...
the dual imposed by the requirement that $C_k$ be part of an optimal basis for the primal. If the vectors in $C_k$ are members of an optimal basis for the primal, then the corresponding constraints of the dual are equalities.

If there exists a feasible basic solution to the constrained dual L.P. problem, (corresponding to the vectors in $C_k$ belonging to (BOB): $B_k$), then we have (BOB): $B_k = [C_k, D_k]$. The vectors in $D_k$ are identified by the non-basic dual slack variables.

If there is no feasible basic solution to the constrained dual L.P. problem we can apply (22), or we can use the objective function: sum of infeasibilities, minimized in Phase I of the simplex method [3], to identify those equality rows with positive shadow prices. At least one of these equality constraints must be relaxed for dual feasibility; i.e., the corresponding set of vectors from $C_k$ cannot together be part of a (BOB). If there are $p_k$ vectors from $C_k$ with positive dual shadow prices when the sum of infeasibilities is a minimum, and if these vectors are identified by $j(+)$, then we can apply the constraint:

$$T^2_k : \sum_{j(+)y_j} \leq p_k - 1$$

(23)

where $y_j$'s are $(0,1)$ variables, to our selection of $C_{k+1}$. In general, (23) will be more restrictive than (22).

Denote the set of constraints of the $T^2$ type, obtained through stage $k$, by $[T^2]_k$. Clearly these apply, in addition to $[T^1]_k$ and (21), to our selection of $C$. 

Given $C_k$ satisfies the constraint sets: $[T^1_k]$, $[T^2_k]$ and (21), and given (BOB): $B^*_k = [C_k, D_k]$, then, if $B^*_k$ is (FBOB), (i.e., if there exists a feasible solution to $P_4$), we can use (14) to define a set $T^1_k$ and a corresponding constraint to add to $[T^1_k]$, to give $[T^1_k]$.

If $B^*_k$ is not (FBOB), we can again use the objective function: sum of infeasibilities, minimized in Phase I of the simplex method of solving $P_4$, to identify the sub-set of those behavioral vectors not belonging to $B^*_k$, (and hence excluded from $P_4$), which could restore feasibility. One or more of these vectors must be a member of subsequent (BOB's) as a necessary (though not sufficient) condition for feasibility.

Let $W_k$ be the basis for problem $P_4$, corresponding to $B^*_k$, for which the sum of infeasibilities is a minimum; and let

$$F_k = [i \mid W_k^{-1}b < 0] \quad (24)$$

Then we define:

$$T^3_k = [H^i_j \mid \sum_{i \in F_k} t^i_j < 0] \quad (25)$$

where

$$t^i_j = W_k^{-1} H^i_j \quad (26)$$

Clearly $\sum_{i \in F_k} t^i_j > 0$ for all $H^i_j \in B^*_k$, since $W_k$ already minimizes the sum of the infeasibilities. The corresponding necessary condition can be represented by the constraint:
\[ \sum_{j} \delta_{x_j} y_j \geq 1 \]  

(27)

where \[ \delta_{x_j} = 1 \text{ if } H_{1j} \in T^2 \]

\[ = 0 \text{ otherwise} \]

and the \( y_j \)'s are \((0, 1)\) variables.\(^{12/}\)

Denote the set of constraints of the \( T^3 \) type obtained at stage \( k \), by \([T^3]_k\). Clearly \([T^3]_k\) apply, in addition to \([T^1]_k\), \([T^2]_k\) and (21), to our selection of \( C_k \).

We may again note that the set of \( H_{1j} \in W_* \), the optimal feasible basis for \( P_5 \), will satisfy all \([T^3]_k\).

Denote the sets of constraints given by \([T^1]_k\), \([T^2]_k\), \([T^3]_k\) and (21), by \([T]_k\). \( C_k \) is now defined as a \((m \times m)\) matrix of vectors from \( H \) satisfying \([T]_k\). The problem of selecting \( C_k \) is discussed in Appendix III.

If at any stage we cannot find \( C_k \) satisfying the constraint set \([T]_k\), then the previous best high-point solution is a global optimum solution to the linear two-level programming problem. This follows since, at this stage, it is impossible to satisfy the necessary conditions for a better high-point solution.

\(^{12/}\) Given a feasible solution to \( P_2 \), and hence \((FBOB): B_* \), we have \( T^3_k \neq \phi \).
3. ALGORITHMIC PRINCIPLES

Since problem P2 can have local optima (and hence there is little hope of continuous monotonic improvement in the objective function), some form of implicit enumeration has to be relied upon, using the global necessary condition information generated in the search for locally better solutions.

The $T^1$ set: $T^1_k$ generated at the high-point in any (FBOB): $B_k$, constitutes a requirement that some subset of the $H_{lj}$ $B_k$ he in any (FBOB) leading to a better high-point value of $f_2$. Hence $B_k$ does not satisfy $T^1_k$. By paying attention to all $T^1$ sets generated to date in the solution process, we can avoid returning to any previously explored (FBOB).

Similarly, by satisfying the constraint sets $[T^2]_k$ and $[T^3]_k$, we can avoid returning to any previously explored set of vectors, e.g., $C_k$, that cannot be part of a (BOB): $B_k = [C_k, D_k]$, or any (ROB) that cannot be made feasible.

These observations lead us to propose the following three broad steps in a solution algorithm for the linear two-level programming problem:

**Step 1:** Attempt to find a set of $(\leq m)$ vectors, $C_k$, from $H_1$, that satisfy the existing $T$-set constraints: $[T]_k$.

**Step 2:** Given $C_k$, attempt to find a behaviorally optimum basis; (BOB): $B_k = [C_k, D_k]$. 
Step 3: Given (BOB): attempt to solve the corresponding policy L.P. problem \((P4)\), to find the high-point in \(B_L\).

If Step 1 cannot be completed, then the best high-point solution obtained to date is the global optimum solution to \(P2\). If Step 2 Cannot be completed, we generate \(T^2\), add the corresponding constraint to \([T]_k\), and return to Step 1. If we cannot find a feasible solution to \(P4\) in Step 3, we generate \(T^3\), add the corresponding constraint to \([T]_k\) and return to Step 1. If \(B_L\) is (FBOB), then from the high-point solution information we generate \(T^1\), add the corresponding constraint to \([T]_k\), to obtain \([T]_L\), and return to Step 1.

Since there exist a finite number of behaviorally optimal bases, and since the constraint set generated at any stage prevents us from returning to the set of vectors used to generate the T-set, the proposed algorithm will find the global optimum solution to the linear two-level programming problem, \(P2\), in a finite number of iterations. The proposed algorithm also examines neighboring bases at the high-point in any (FBOB): for the opportunity of making at least some monotonic improvement in the value of \(f_2\), while using only the \(T^1\) set necessary conditions.

1. Set \(k = 1\), and \(f_2^k = -\infty\), then attempt to solve \(P5\). If \(P5\) is infeasible, go to Step 2. If \(P5\) is unbounded, go to Step 3. Otherwise go to Step 4.
2. Since \( P5 \) is infeasible, so is problem \( P2 \). Stop.

3. Impose an arbitrary upper limit on \( f_2 \), and go to Step 1.

4. The solution to \( P5 \) provides an upper limit on \( f_2 \). Given the solution values for the policy variables from Step 1, \( x_2 = ^* \), solve \( P3 \). Denote the optimal feasible basis: \( B_k \). If \( P3 \) is unbounded, place an arbitrary upper limit on \( f_1 \) and go to Step 4. Otherwise go to Step 9.

5. Set \( k = k + 1 \) and use Appendix III to select a set of vectors \( C_k \) which satisfy \( [T]_k \). If this is impossible, go to Step 6; otherwise go to Step 7.

6. The implicit search is complete, and \( f_2^* \) is the optimal value of the policy objective function for \( P2 \); \( B_* \) and \( W_* \) are the corresponding optimal bases. Stop.

7. Find (BOB): \( B_k = [C_k, D_k] \) and go to Step 9. If this is not possible, go to Step 8.

8. Generate \( T_k^* \) as in (22) and add the corresponding constraint to \( [T]_k \). Go to Step 5.

9. Attempt to find a feasible solution to \( P4 \), given (BOB): \( B_k \), by minimizing the sum of infeasibilities, (phase I, simplex method). If \( P4 \) is feasible, go to Step 10. If \( P4 \) is infeasible, define \( T_k^3 \) as in (25), add this to the set of constraints \( [T]_k \), and go to Step 5.
10. Find the optimum solution to $P_k$. Define $T_k^1$ according to (14). If $T_k^1 = \phi$, go to Step 6, otherwise add $T_k^1$ to the set of constraints $[T]_k$, and go to Step 11.

11. Construct $R_k$ as in (18) and calculate $d_{ij}$ as in (19). For each $i \in R_k$, identify the neighboring vector $H_{1r}$ (should one exist), by (20), and check whether $H_{1r} \in T_k^1$. As soon as a neighboring vector belonging to $T_k^1$ is found, go to Step 14. If there are no neighboring vectors belonging to $T_k^1$, go to Step 12.

12. If $f_2^*(k) < f_2^*$, go to Step 5. Otherwise go to Step 13.

13. Set $f_2^* = f_2^*(k)$, $B_* = B_k$, $W_* = W_k$; go to Step 5.

14. Set $k = k + 1$. Obtain a better neighboring basis by substituting $H_{1r}$ for $H_{1i}$. Call the new basis $B_k$, and go to Step 10.

5. REMARKS

Steps 1, 2 and 3. These steps initialize the algorithm, test for feasibility, and place an upper bound on $f_2$.

Step 4. Since $x_2 = x_2(k)$ yields a feasible solution to $P_5$, it also yields a feasible solution to $P_3$ since the restraints are the same. If $P_3$ is unbounded we place an arbitrary upper limit on $f_1$ and repeat. If $P_3$ is bounded, then a behaviorally optimal basis (HOB), $B_k$, has been found. We go to Step 9. This completes the initialization.
**Step 5.** Formally we have a combinatorial problem of selecting a set of integers (corresponding to the selection of a set of vectors $C_k$ from $H_1$) satisfying the $T$-set constraints, $[T]_k$.

As remarked previously, the $H_{1j} \in W_*$, the solution obtained in Step 1, will satisfy $[T^1]_k$ and $[T^2]_k$. Thus it is only when the number of restraints $[T^2]_k$ is greater than the number of vectors $H_{2q} \in W_*$, that a non-trivial problem arises. Non-trivial problems can be solved using the algorithm discussed in Appendix III.

**Step 6.** The inability to solve the combinatorial problem indicates that a set of mutually inconsistent conditions has been arrived at, and the search of the solution space is complete. There is no set of vectors $C_k$ which satisfy the necessary conditions for a better high-point solution.

**Step 7.** Attempts to find a (ROB): $B_k$ that includes $C_k$, i.e., satisfies all $T$-set constraints, lie can either use a modified simplex method or formulate a constrained dual L.P. problem.

**Step 8.** If $C_k$ does not form part of a (BOB) we can define a new $T_2$-set constraint, limiting future $C$ matrices to at most $m_{1-1}$ vectors from $C_k$.

**Step 3.** Starting with (BOB): $B_k$, we define $P_4$ and attempt to find a feasible solution by the choice of $x_2$. If this is impossible, we construct a $T_3$ set corresponding to the necessary conditions for a feasible solution to the policy L.P. problem.
Step 10. Given (FBOB) : \( B_k \), we find the high-point solution and define \( T_k^1 \).

If \( T_k^1 \) is empty, then our high-point solution is an optimal feasible solution to \( P_2 \) (since it is also an optimal feasible solution to \( P_5 \)). If \( T_k^1 \) is not empty, we add the corresponding T-set constraint to \([T]_k^1\).

Step 11. Check for the existence of a better neighboring basis to \( B_k \), at the high-point setting for the policy variables obtained in Step 10.

Step 12. If there are no better neighboring bases to \( B_k \), we check whether the high-point value for \( f_2 \) is the best obtained to date.

Step 13. Records the best high-point solution to date.

Step 14. Makes a basis change to a better neighboring basis with high-point value for \( f \geq f^{x(k)}_2 \). Since this neighboring basis is (FBOB) by construction, the algorithm returns to Step 10.

6. NUMERICAL EXAMPLE

As a specific numerical example of the application of the algorithm, consider the problem:
Find $x_{1j}$ $(j = 1, \ldots, 6)$ and $x_{2q}$ $(q = 1, 2)$ such that:

$$f_2 = \max_{x_{2q}} \left( -\frac{1}{2}x_{11} + 40x_{12} + 4x_{13} + 8x_{21} + 4x_{22} \right)$$  \hspace{1cm} (28)$$

subject to

$$f_1 = \max_{x_{1j}} \left( -x_{11} - x_{12} - 2x_{13} - x_{21} - 2x_{22} \right)$$  \hspace{1cm} (29)$$

and

$$H_1 x_1 + H_2 x_2 = b$$  \hspace{1cm} (30)$$

$$x_1, x_2 > 0$$  \hspace{1cm} (31)$$

where

$$H_1 = \begin{bmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 2 & -0.5 & 0 & 1 & 0 \\ 2 & -1 & -0.5 & 0 & 0 & 1 \end{bmatrix}$$$$H_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

A contour map of $f_2$ for this problem in the "policy space" $(x_{21}, x_{22})$ is given in Figure 1.

Applying the algorithm:

**Step 1.** We solve problem $P_5$, i.e., we maximize $f_2$ with respect to all $x$'s subject to (30) and (31). This yields the first simplex tableau
Figure 1: POLICY OBJECTIVE FUNCTION CONTOURS GIVEN OPTIMAL FEASIBLE BASES FOR NUMERICAL EXAMPLE
### Table 1: Simplex Tableaus Used in Solution of Numerical Example

<table>
<thead>
<tr>
<th>n = 1</th>
<th>b</th>
<th>H_{14}</th>
<th>H_{15}</th>
<th>H_{16}</th>
<th>H_{11}</th>
<th>H_{12}</th>
<th>H_{13}</th>
<th>H_{21}</th>
<th>H_{22}</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>H_{11}</td>
<td>1.5</td>
<td>.5</td>
<td>.17</td>
<td>.83</td>
<td>1</td>
<td></td>
<td></td>
<td>.33</td>
<td>1.67</td>
<td>Optimum solution to P5 (f_{2j} &gt; 0, f_{2q} &gt; 0).</td>
</tr>
<tr>
<td>H_{12}</td>
<td>1.5</td>
<td>.5</td>
<td>.5</td>
<td>.5</td>
<td>1</td>
<td></td>
<td></td>
<td>1</td>
<td>1</td>
<td>Upper bound on f_2 = 58.</td>
</tr>
<tr>
<td>H_{13}</td>
<td>1</td>
<td>1</td>
<td>- .33</td>
<td>.33</td>
<td>1</td>
<td></td>
<td></td>
<td>- .67</td>
<td>.67</td>
<td></td>
</tr>
<tr>
<td>Z - C_1</td>
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<td>-3</td>
<td>0</td>
<td>-2</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>- 2</td>
<td></td>
</tr>
<tr>
<td>Z - C_2</td>
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<td>22</td>
<td>18</td>
<td>18</td>
<td></td>
<td></td>
<td></td>
<td>28</td>
<td>32</td>
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<th>H_{15}</th>
<th>H_{16}</th>
<th>H_{11}</th>
<th>H_{12}</th>
<th>H_{13}</th>
<th>H_{21}</th>
<th>H_{22}</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>H_{14}</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td>BOB (f_{1j} &gt; 0)</td>
</tr>
<tr>
<td>H_{15}</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td>-1</td>
<td>2</td>
<td></td>
<td>-.5</td>
<td>2</td>
<td>Bases which by inspection, are also FBOB: [H_{14} H_{11} H_{16}]</td>
</tr>
<tr>
<td>H_{16}</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td></td>
<td>-.5</td>
<td>2</td>
</tr>
<tr>
<td>Z - C_1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
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<td>High Point ( f_{2j} &gt; 0 ) ( H_{1j} \in B_1 )</td>
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<td>.5</td>
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Table 1 (cont.d)  Simplex Tableaus Used in Solution of Numerical Example

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<td>( H_{12} )</td>
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<td>( H_{12} \in B_2, f_{22} \geq 0 ).</td>
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<td>( H_{22} )</td>
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<td>.5</td>
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<td>.67</td>
<td>-.67</td>
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<td>for ( x_{21} = 0, x_{22} = .75 )</td>
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<td>Unbounded, unless T-sets violated. Since no positive pivot for $H_{15}$.</td>
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in Table 1, and provides an upper bound of $58.0$ on $f_2$. In Table 1, $b$ is the right hand side and $H_{14}, H_{15}$ and $H_{16}$ are the "disposal" activities. Since the solution to problem $P5$ is feasible and bounded, we go to Step 4.

**Step 4:** In tableau $n = 1$ of Table 1, $x_{21} = x_{22} = 0$. Fixing these values of the policy variables, $x_{21}$ and $x_{22}$, we solve $P3$ to get a BOB. This solution value is shown in tableau $n = 2$ of Table 1.

(BOB): $B_1 = [H_{14}, H_{15}, H_{16}]$. Since a EOG has been found, go to Step 9.

**Step 9:** The solution obtained in Step 4 is already FBOB, go to Step 10.

**Step 10:** We find the optimum solution to $P4$ for $B_1$. This optimal solution is given in tableau $n = 3$ of Table 1. The only $f_{2j} < 0$ in tableau $n = 3$ is for $H_{12}$, hence $T_1 = [H_{12}]$, and we know that any better basis must contain $H_{12}$. Go to Step 11.

**Step 11:** $R_1 = [H_{15}, H_{16}]$ since the levels of these vectors in $B_1$ have been driven to zero. For row 2 of tableau $n = 2$ (i.e., for the BOB tableau) we have:

$$d_{21} = -rac{1}{1} = -1$$

$$d_{23} = -rac{2}{5} = -0.4$$

$$d_{21} = \min d_{2j}, H_{12} \notin T_1$$

For the BOB tableau, we have:

$$d_{21} = -rac{1}{1} = -1$$

$$d_{23} = -rac{2}{5} = -0.4$$

$$d_{21} = \min d_{2j}, H_{12} \notin T_1$$
and for row 3, we have:

\[ d_{32} = \frac{-1}{-1} = 1 \]

\[ d_{33} = \frac{-2}{-0.5} = 4 \]

\[ d_{32} = \min_j d_{ij} H_{12} \in T_1 \]

Since \( H_{12} \in T_1 \), we know that there is a better neighboring (FBOB) at \( d_2 = \frac{2}{2} = 1 \), \( x_{21} - x_{22} = 0.5 \). We go to Step 14.

Step 14. Set \( k = k+1 = 2 \). Define a new basis \( B_2 = B_1 + H_{12} - H_{16} = [H_{14}, H_{15}, H_{12}] \). The corresponding (FBOB) is given in tableau \( n = 4 \) of Table 1. This basis is not Feasible in tableau \( n = 4 \), but by construction we know it would be feasible for \( x_{21} - x_{22} = 0.5 \). Go to Step 10.

Step 10. Solution of problem \( P_4 \) for \( B_2 \) leads to tableau \( n = 4 \) in Table 1. The new T-set is \( T_2 = [H_{11}, H_{13}] \). (At this stage we know that any better basis has \( H_{12} \) (from \( T_1 \)) and "either \( H_{11} \) or \( H_{13} \) or both" (from \( T_2 \)) ). Go to Step 11.

Step 11. \( R_2 = [H_{15}] \) since only this member of \( B_2 \) has been driven to zero. For row 2 of tableau \( n = 4 \) (i.e., for the BOB tableau) we have

\[ d_{23} = -1.5 = \min_j d_{2j} \]
and \( H_{13} \in T_2 \).

Since \( H_{13} \in T_2 \), we know that there is a "better" neighboring (FBOB) at \( x_{21} = 0 \), \( x_{22} = 0.75 \). We go to Step 14.

**Step 14.** Set \( k = k + 1 = 3 \). Define a new basis \( B_3 = B_2 + H_{13} - H_{15} - \{H_{14}, H_{12}, H_{13}\}\). The corresponding (FBOB) is given in tableau \( n = 6 \) of Table 1. This tableau is not feasible, but since it is a neighboring basis to an (FBOB), it must itself be feasible for appropriate values of the policy variables. Go to Step 10.

**Step 10.** Solution of problem \( P_4 \) for \( B_3 \) leads to tableau \( n = 7 \) in Table 1. The new \( T \)-set is \( T_3 = [H_{11}] \) and we see that \( T_3 \) dominates \( T_2 \). We now know that any better basis contains \( H_{11} \) (from \( T_3 \)) and \( H_{12} \) (from \( T_1 \)). Go to Step 11.

**Step 11.** \( R_3 = [H_{14}] \). Since the \( H_{14} \) row in tableau \( n = 6 \), (the BOB tableau) has no negative elements, ratio \( d_{ij} \), and hence \( \min d_{ij} \), are not defined, and we know that there is no better neighboring basis. (Indeed no better adjacent basis.) Go to Step 12.

**Step 12.** \( f_2^* = 29.2 > f_2 = -\infty \), hence go to Step 13.

**Step 13.** \( \bar{f}_2 = f_2^*(3) = 29.2 \), \( B_4 = [H_{14}, H_{12}, H_{13}] \) and \( W_2 = [H_{12}, H_{13}, H_{22}] \). Go to Step 5.
Step 5. Set \( k = k + 1 = 4 \). We revert to basis \( B_4 = [H_{11}, H_{12}, H_{13}] \) which satisfies \( T_1^1 \) and \( T_3^1 \) (and hence \( T_2^1 \)) and is recorded in tableau \( n = 1 \). Go to Step 7.

Step 7. From the first tableau in Table 1, we carry out a dual pivot step (where \( H_{11} \) and \( H_{12} \) are not considered candidates for removal, since this would violate the T-sets), to replace \( H_{13} \) by \( H_{16} \) in the basis. This results in tableau \( n = 8 \), where \( f_{12} < 0 \), but only positive pivots would remove \( H_{11} \) or \( H_{12} \) from the basis (thus violating the T-sets). Thus a basis containing both \( H_{11} \) and \( H_{12} \) cannot be made BOP. Thus \( T_4^2 = [H_{11}, H_{12}] \). Go to Step 8.

Step 8. We have that \( H_{11} \) and \( H_{12} \) cannot together be in a (BOP). The full set of T constraints at this stage is given below, where \( y_j \)'s are (0, 1) variables.

\[
\begin{align*}
y_1 & \quad y_2 & \quad y_3 & \quad y_4 & \quad y_5 & \quad y_6 \\
T_1^1 & 1 & \geq 1 \\
T_2^1 & 1 & 1 & \geq 1 \\
T_3^1 & 1 & \geq 1 \\
T_2^1 & 1 & 1 & \leq 1 \\
\text{Basis} & 1 & 1 & 1 & 1 & 1 & 1 & \leq 3
\end{align*}
\]
It is easily seen in this small example that there is no feasible solution to this set of constraints, i.e., there is no set of vectors from \( H_1 \) that satisfy the existing T-set necessary conditions for a high-point solution. Go to Step 6.

Step 6. \( f_2 = 29.2, \quad B_* = [H_{14}, H_{12}, H_{13}] \), \( W_* = [H_{12}, H_{13}, H_{23}] \)

Stop.

As can be seen from Figure 1, there is another local optimum at \( x_{21} = 1.5, \quad x_{22} = 0 \), but the information contained in the T-sets is sufficient to rule out any need to visit this lower local optimum.

7. CONCLUSION

The above algorithm allows an implicit search of the policies in problem \( P_2 \), which affect the right hand side, or resource availabilities for the behavioral problem. It does not apply to the many government policies which operate by affecting prices, and hence the objective function of the inner, behavioral, problem. We have not yet done any work on this policy problem, other than to note that it exists and is, in some sense, the "dual" of the problem considered in this paper. Ideally an algorithm will eventually be presented which allows both the objective function and restraints of the behavioral problem to be influenced by policy makers.

Given the crucial role played by the T-sets in the above algorithm, it may be conjectured that an algorithm which focused primarily on the tightness of the T-sets which could be generated, would hold promise of a substantial improvement over the algorithm offered here.
A small FORTRAN computer code has been written to implement the above algorithm for programs where the updated tableau can be carried explicitly. No progress has yet been made in incorporation of the algorithm into a large code. Considerable success has, however, been experienced with a heuristic search of a linear two-level programming problem consisting of 53 constraint rows, 314 behavioral activities (excluding slacks) and 8 policy activities. This heuristic search procedure, based on the above ideas, is discussed in Appendix II.
REFERENCES


Appendix I: Induced Degeneracy at a High Point

Assume \( x_2 = x_2 \) is a unique optimal setting for the policy variables in the two-level programming problem \( P2 \). Let \( B^* \) be an optimal feasible basis for the behavioral L.P. problem \( P3 \) when \( x_2 = x_2 \). From the definition of \( P2 \) it follows that \((x_1^*, x_2^*)\) is a feasible optimal solution to the two-level programming problem \( P2 \), where:

\[
x_1^* = B_1^{-1} b - B_2^{-1} H_2 x_2^* \quad \text{for } H_{1j} \in B^*
\]

\[
= 0 \quad \text{otherwise.}
\]

Let \( W^* \) be an optimal feasible basis for the policy L.P. problem \( P4 \) corresponding to \( B^* \).

We now make the following partitions:

\[
H_1 = [B^*_x : H_{1h}]
\]

\[
= [B^*_x w : B^*_x h : H_{1h}]
\]

\[
H_2 = [H_{2w} : H_{2h}]
\]

\[
W^* = [B^*_w : H_{2w}]
\]

where \( B^*_x \) is \((m \times m)\), \( W^*_x \) is \((n \times m)\), \( B^*_w \) is \((m \times n_1)\), \( H_{2w} \) is \((m \times m_2)\) and \( m_1 + m_2 = m \).

The solution values for the policy L.P. problem \( P4 \) corresponding to are obtained by solving:

\[
B^*_w x_1^* + H_{2w} x_2^* = b
\]

to obtain:

\[
\begin{bmatrix}
  x_1^* \\
  \vdots \\
  x_2^*
\end{bmatrix} = W^{-1}_w b
\]

(I.1)
Thus, if $x_2$ is unique, there can be at most $m$ non-zero values $x_{2q}$, and these are given by $x_{2}^{**}$. (This does not rule out the possibility of a degenerate optimal basic solution to $P_4$ where some $x_{2q}^{**}$ values equate zero.)

At the high-point (in $B_*$) setting for the policy variables, the constraint set for the behavioral L.P. problem is:

$$H_1 x_1 = b - H_2 x_2^{*}$$
$$= b - H_{2w} x_{2}^{**}$$

since, as we have just shown, $x_{2q}^{*} = 0$ for $x_{2q}^{**} \neq x_2^{**}$.

Multiply both sides of (I.2) by $W_{1}^{-1}$ to obtain:

$$W_1^{-1} H_1 x_1 = W_1^{-1} b - W_1^{-1} H_{2w} x_{2}^{**}$$

Since, as we have just shown, $x_{2q}^{*} = 0$ for $x_{2q}^{**} \neq x_2^{**}$.

Multiply both sides of (I.2) by $W_1^{-1}$ to obtain:

$$W_1^{-1} H_1 x_1 = W_1^{-1} b - W_1^{-1} H_{2w} x_{2}^{**}$$

$$= \begin{bmatrix} x_{1}^{**} \\ \vdots \\ x_{2}^{**} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ x_{2}^{**} \end{bmatrix}$$

$$= \begin{bmatrix} x_{1}^{**} \\ \vdots \\ 0 \end{bmatrix}$$

This result follows from (I.1) and $W_1^{-1} W_* = [W_1^{-1} B_{2w} ; W_1^{-1} H_{2w}]$

$$= \begin{bmatrix} I_{m_1} & 0 \\ - & 0 \\ 0 & I_{m_2} \end{bmatrix}$$
Now, multiply both sides of (h.3) by $W$:

$$H_1 x_1 = [B_{xW} : H_{2W}] \begin{bmatrix} x_1^* \\ \vdots \\ 0 \end{bmatrix}$$

$$H_1 x_1 = B_{xW} x_1^* \begin{bmatrix} x_1^* \\ \vdots \\ 0 \end{bmatrix}$$

$$= [B_{xW} : B_{xh}] \begin{bmatrix} x_1^* \\ \vdots \\ 0 \end{bmatrix}$$  \hspace{1cm} (I.4)

To obtain the solution values to the behavioral L.P. problem $P_3$, corresponding to $B^*_x$, we multiply the right-hand side of (I.4) by $B_{xh}$, to obtain:

$$x^*_1 = \begin{bmatrix} x_1^* \\ \vdots \\ 0 \end{bmatrix}$$  \hspace{1cm} \text{for } H_{1j} \in B^*_x$$

$$= 0 \hspace{1cm} \text{otherwise.}$$  \hspace{1cm} (I.5)

From (I.5) we have the result that, at the high-point setting for the policy variable in (EOB): $B^*_x$, the optimal feasible solution to the behavioral L.D. problem $P_3$, corresponding to $B^*_x$, will be degenerate if at least one vector from $H_2$ is a member of the optimal feasible basis corresponding to the high-point in $B^*_x$.
Appendix II: Heuristic Procedure for Finding a Locally Better High-point Solution to the M.L.I. Problem. (Two-level, linear problem. Modified steepest ascent.)

**STEP 1** Solve $\mathcal{P}_5$ to obtain a feasible setting for the policy variables, $x_2 = x_2^{(1)}$, and an upper bound on $f_2^*$, $(f_2^*)$.

**STEP 2** Solve $\mathcal{P}_3$ given $x_2 = x_2^{(1)}$, to obtain (FBOB): $B_1$.

**STEP 3** Solve $\mathcal{P}_4$ given (BOB): $B_1$, to obtain the high-point setting for the policy variables: $x_2 = x_2^{(1)}$. If $f_2^{*(1)} < f_2^{**}$, continue.

**STEP 4** Construct the vector:

$$y^{(1)} = [x_2^{(1)} - x_2^{(1)}]$$

Consider the directed line segment; $\theta y^{(1)}$, from the point $x_2^{(1)}$. Thus:

$$x_2^{(2)} = x_2^{(1)} + \theta y^{(1)},$$

where $\theta > 0$, such that $x_2^{(2)} \geq 0$.

The value for $\theta$ is subject to choice, subject to the above constraints.

For $x_2^{(2)} \neq x_2^{(1)}$, proceed to STEP 5.

For $x_2^{(2)} = x_2^{(1)}$, proceed to STEP 6.

**STEP 5** Attempt to solve $\mathcal{P}_3$ given $x_2 = x_2^{(2)}$, as defined in (II-1).

(a) **No feasible solution**: reduce the value for $\theta$ so that $x_2^{(2)}$ approaches $x_2^{(1)}$; return to start of STEP 5. (This loop is constrained to a finite number of steps before $x_2^{(2)} = x_2^{(1)}$, and we go to STEP 6.)

(b) **Feasible optimal solution** and $f_2^{*(2)} < f_2^{*(1)}$, for the associated $(x_1, x_2)$ solution values. Reduce the value for $\theta$ as in (a).
(c) Feasible optimal solution and $f_2^{(2)} > f_2^{*(1)}$, for the associated $(x_1, x_2)$ solution values. Update the (FBOB) in STEP 2 to the current (FBOB), and return to STEP

STEP 6 Whenever $x_2^{(k)} - x_2^{(k-1)}$, we conduct an experiment around $x_2^{(k-1)}$, i.e., in 6-neighborhood of $x_2^{(k-1)}$. At each experimental point we attempt to solve P3, (i.e., STEP 2), and hence evaluate $f_2 = f_2^k - f_2^{*(k-1)}$.

We either find $f_2 > 0$ in some direction in which case we go to STEP 2, or we conclude $x_2^{(k-1)}$ is a local optima.

Comments

This heuristic search has been used to find an advanced starting position (hi&!!-point solution and T-set), for a linear two-level programming problem consisting of 53 constraint rows, 314 behavioral activities (excluding slacks) and 8 policy activities. The computer program was written by Richard Inman. A summary of solution values, and steps, is given below.
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<thead>
<tr>
<th>Trial</th>
<th>Problem</th>
<th>$f_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1)$</td>
<td>$x_2$ : $20.0$</td>
<td>121.4</td>
</tr>
<tr>
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<td>-</td>
<td>636.6</td>
</tr>
<tr>
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<td>-</td>
<td>$50,235$</td>
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<tr>
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<td>$x_2$ : $20.0$</td>
<td>145.6</td>
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<td>$x_2$ : $20.0$</td>
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<td>313.3</td>
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<td>-</td>
<td>$32,982$</td>
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<tr>
<td>$(4)$</td>
<td>$x_2$ : $20.0$</td>
<td>153.1</td>
</tr>
<tr>
<td>$x_2$ : $20.0$</td>
<td>-</td>
<td>19.4</td>
</tr>
<tr>
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<td>-</td>
<td>$43,666$</td>
</tr>
<tr>
<td>$(5)$</td>
<td>$x_2$ : $20.0$</td>
<td>153.3</td>
</tr>
<tr>
<td>$x_2$ : $20.0$</td>
<td>-</td>
<td>9.7</td>
</tr>
<tr>
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<td>-</td>
<td>Infeasible</td>
</tr>
<tr>
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<td>$x_2$ : $20.0$</td>
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</tr>
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<td>-</td>
<td>247.9</td>
</tr>
<tr>
<td>$x_2$ : $20.0$</td>
<td>-</td>
<td>$43,466$</td>
</tr>
<tr>
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<td>154.0</td>
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<tr>
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<td>247.5</td>
</tr>
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<td>-</td>
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</tr>
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<td>-</td>
<td>123.7</td>
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<td>-</td>
<td>$43,466$</td>
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<tr>
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<td>169.1</td>
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<td>-</td>
<td>8.9</td>
</tr>
<tr>
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<tr>
<td>$(13)$</td>
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</tr>
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<td>15.7</td>
</tr>
<tr>
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<td>-</td>
<td>$49,905$</td>
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<tr>
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<td>168.8</td>
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<td>$x_2$ : $20.0$</td>
<td>-</td>
<td>8.9</td>
</tr>
<tr>
<td>$x_2$ : $20.0$</td>
<td>-</td>
<td>$49,905$</td>
</tr>
</tbody>
</table>

* Same basis as obtained two steps earlier, therefore experiment about $x_2^{(11)}$. 
The results in the first row of Table AII.1 are obtained by solving P5 to give an upper bound of 50,235 on \( f_2 \) and the initial setting for the policy variables: \( x_2^{(1)} \). For the subsequent rows of this table, we have:

Row 2: Solve \( P_3 \) given \( x_2^{(1)} \rightarrow B_1 \).

Row 3: Solve \( P_4 \) given \( x_2^{(2)} \rightarrow B_2 \). Form \( y_1 = [x_2^{(2)} - x_2^{(1)}] \).

Row 4: Solve \( P_3 \) given \( x_2^{(3)} = x_2^{(2)} + \theta_1 y_1 \rightarrow B_2 \).

Row 5: Solve \( P_4 \) given \( B_2 \rightarrow x_2^{(4)} \). Form \( y_2 = [x_2^{(4)} - x_2^{(3)}] \).

Row 6: No feasible solution to \( P_3 \) given \( x_2^{(5)} = \frac{1}{2} x_2^{(4)} + \theta_2 y_2 \).

Row 7: The present program sets \( \theta_2 = 0 \) and solves \( P_3 \) given \( x_2^{(4)} \rightarrow B_3 \), \( B_3 \neq B_2 \), i.e. an adjacent basis to \( B_2 \) has been found at \( x_2^{(4)} \), therefore continue.

Row 8: Solve \( P_4 \) given \( B_3 \rightarrow x_2^{(6)} \). At this point the (arbitrary) limit on the number of iterations for this trial run was reached.

Row 9: We chose to restart the search procedure by solving \( P_3 \) given \( x_2^{(6)} \rightarrow B_4 \). We were lucky since \( B_4 \neq B_3 \). If we had obtained \( B_6 = B_3 \), we would have calculated a probe based on \( [x_2^{(6)} - x_2^{(4)}] \) - which is what the program would have done if the iteration limit had not been reached.

Row 10: Solve \( P_4 \) given \( x_2^{(7)} \rightarrow x_2^{(6)} \). Form \( y_3 = [x_2^{(7)} - x_2^{(6)}] \).

Row 11: Solve \( F_3 \) given \( x_2^{(8)} = x_2^{(7)} + \theta_3 y_3 \rightarrow v_5 \).
Appendix III: Finding a Set of Vectors from $H_1$ that Satisfy $[T]_k$

For the purpose of discussion assume $[T]_k$ consists of $r$ constraints of the $T^1$-type, $s$ constraints of the $T^2$-type and $t$ constraints of the $T^3$-type. To find at most $m$ vectors from $H_1$ which satisfy $[T]_k$, we need to solve:

P6 Find $y_1, \ldots, y_{n_1}$ such that:

\[ \sum_{j} y_j \leq m \]  \hspace{1cm} (III-1)

\[ [T^1]_k: \sum_{j} \delta_{ij}^1 y_j \geq 1 \quad i = 1, \ldots, r \]  \hspace{1cm} (III-2)

\[ [T^2]_k: \sum_{j} \delta_{ij}^2 y_j \leq p_i - 1 \quad i = 1, \ldots, s \]  \hspace{1cm} (III-3)

\[ [T^3]_k: \sum_{j} \delta_{ij}^3 y_j \geq 1 \quad i = 1, \ldots, t \]  \hspace{1cm} (III-4)

\[ y_j - y_j^2 = 0 \quad j = 1, \ldots, n_1 \]  \hspace{1cm} (III-5)

Where $\delta_{ij}^u = 1$ if $H_{ij} \in T_i^u$, $= 0$ otherwise.

This is an integer programming problem, i.e., $y_j$'s are $(0, 1)$ variables as given by (III-5), for which any feasible solution is acceptable, [5].
Feasible solutions to $P_6$ will be members of the set of feasible solutions to:

$P_7$ Find $y_1, \ldots, y_{n_1}$ such that

$$0 \leq y_j \leq 1 \quad j = 1, \ldots, n_1$$

(III-6)

and $(I-), (III-2), (III-3), (III-4)$.

If $P_7$ is infeasible, then so is $P_G$.

$P_7$ is a linear programming problem without an objective function. Any objective function can be added to $P_7$ which will help identify a feasible solution to $P_G$.

As an initial objective function we can use

$$z^1 = \sum y_j \quad \text{2 minimum}$$

(III-7)

If the solution to $P_7, y^k$, which minimizes (II-7) does not yield integer values of $y_j$, we can replace (III-7) by:

$$z^k = \sum c_j^k y_j \quad \text{a minimum}$$

(III-8)

where, $c_j^k = 1$ for $0 \leq y_j^k < .5$

(III-9)

$c_j^k = -1$ for $.5 < y_j^k \leq 1$.
and for \( y_j^k = .5 \), \( c_j^k = +1 \) or \(-1\), at random.

Hence, for example, if \( y_j^k < .5 \), \( c_j^k = 1 \) and since we minimize \( z^k \), we would expect \( y_j^{k+1} \) to tend towards zero.

Since we seek an objective function that will accelerate the search for a feasible solution to \( P_7 \) such that \( \sum_j (y_j - y_j^2) = 0 \), and therefore a feasible solution to \( P_6 \), the above suggestions would appear sufficient.

We may also note that \( P_6 \) can be converted into a weaker set covering problem \([\ ]\): \( P_8 \)

Find \( y_1, \ldots, y_n \) such that:

\[
\begin{align*}
\sum_j y_j & \quad \text{a min} \\
st & \sum_j \phi_{ij}^2 y_j \geq 1 \quad i = 1, \ldots, s \\
\end{align*}
\]

and \((\text{III-2}), (\text{III-4})\) and \((\text{III-5})\).

Where \( \phi_{ij}^2 = 0 \) if \( H_{ij} \in T_1^2 \)

\( = 1 \) otherwise.

Inequality \((\text{III-11})\) is derived from \((\text{III-3})\). Since the full basis must contain \( m \) vectors, it follows from \((\text{III-3})\) that:

\[
\sum_j \phi_{ij}^2 y_j \geq m+1-r_i \geq 1
\]

The authors have not had extensive experience with the application of problem \( P_6 \). In the few cases we have examined, \( P_7 \) has simply been infeasible, thereby proving \( P_6 \) is also infeasible.