Johannes Bisschop, Wilfred Candler, John H. Duloy, and Gerald T. O'Mara

The Indus Basin Model: A Special Application of Two-Level Linear Programming

World Bank Reprints


No. 203. Peter T. Knight, "Brazilian Socioeconomic Development: Issues for the Eighties,"  *World Development*

No. 204. Hollis B. Chenery, "Restructuring the World Economy: Round II,"  *Foreign Affairs*

No. 205. Uma Lele and John W. Mellor, "Technological Change, Distributive Las, and Labor Transfer in a Two Sector Economy,"  *Oxford Economic Papers*

No. 206. Gershom Feder, "Adoption of Interrelated Agricultural Innovations: Complementarity and the Impacts of Risk, Scale, and Credit,"  *American Journal of Agricultural Economics*


No. 208. Michael Cernea, "Indigenous Anthropologists and Development-Oriented Research,"  *Indigenous Anthropology in Non-Western Countries*


No. 210. George Psacharopoulos, "Returns to Education: An Updated International Comparison,"  *Comparative Education*

No. 211. Gregory K. Ingram and Alan Carroll, "The Spatial Structure of Latin American Cities,"  *Journal of Urban Economics*


No. 213. Salah El Seraty, "Absorptive Capacity, the Demand for Revenue, and the Supply of Petroleum,"  *Journal of Energy and Development*


No. 215. Michael Cernea, "Modernization and Development Potential of Traditional Grass Roots Peasant Organizations,"  *Directions of Change: Modernization Theory, Research, and Realities*

No. 216. Avishek Braverman and T. N. Srinivasan, "Credit and Sharecropping in Agrarian Societies,"  *Journal of Development Economics*

THE INDUS BASIN MODEL: A SPECIAL APPLICATION OF TWO-LEVEL LINEAR PROGRAMMING

Johannes BISSCHOP, Wilfred CANDLER, John H. DULOY and Gerald T. O’MARA*

Development Research Center, The World Bank, Washington, DC 20433, U.S.A.

Received 24 January 1980
Revised manuscript received 26 May 1981

A basic and verbal description of the Indus Basin Model is presented. The model is an example of a strategic planning exercise designed to aid in the specification of surface and ground water related policies in Pakistan. It is also a special application of the two-level linear programming problem. The concept of multi-level programming is introduced, and the general two-level linear program is described as a non-convex problem. It is shown, however, that linear programming techniques can be used in the particular application of the Indus Basin Model.

Key words: Application, Multi-level Programming, Linear Programming.

1. Introduction

This paper is centered around an ongoing large-scale modeling effort designed to aid in the specification of surface and ground water policies in Pakistan. Rather than giving the reader a detailed exposition of the model and some of its early results (a formidable task by itself), we have chosen to limit our model description to a minimum, and concentrate instead on one aspect which, we feel, is often ignored or obscured in similar applications. This is the problem of hierarchical decision making [4, 5, 6]. Economic models designed for policy analysis usually involve two kinds of agents: policy makers and policy receivers. If the policy receivers are optimizing agents, one is faced with a hierarchical decision making problem, or equivalently, a multi-level programming model. In the case of the Indus Basin Model, the government plays the role of the policy maker, while the farmers play the role of policy receivers. The government decides on surface water allocations, and sets taxes and/or subsidies. The farmers, in turn, react to the setting of these policy instruments by using water (both surface and ground water) and choosing cropping patterns so as to maximize their own net income. As the response surface of policy receivers is, in general not necessarily convex, an overall problem involving both types of agents may be a non-convex programming problem. In that event, remedies such as using a weighted combination of the two objective functions will result in

* Views expressed are those of the authors and do not necessarily reflect those of the World Bank or its affiliated organizations.
meaningless solutions. As it turns out, however, linear programming techniques can be used in the case of the Indus Basin problem. This is discussed in Section 4. First, however, we will provide a verbal description of the Indus Basin Model in Section 2, followed by a generic introduction to the concept of multi-level programming in Section 3. The paper concludes with the description of some numerical experiments in Section 5.

2. The Indus Basin Model Family

Following the partition into the nations of India and Pakistan in 1947, India diverted for its own use some of the water from those rivers that formerly fed into parts of the irrigation system in what is now Pakistan. An international crisis emerged, which was finally resolved with the Indus Waters Treaty of 1960. Today, the water supply in Pakistan comes mainly from the river Indus and its westermost tributaries. Control of the water from the eastern tributary rivers is retained by India. The Treaty of 1960 resulted in replacement works consisting of two large reservoirs, three major barrages, 400 miles of new link canals for transferring water to affected areas, some remodeling of existing link canals and barrages, and a program of tubewells and drainage. Although these investments have all been made, there is a continuous demand for further extensions and modifications to the existing water supply system.

The artificial redistribution of water is costly, and requires joint management of ground and surface water. In many areas of the Indus Basin, for instance, tubewell development has not occurred, so that sustained application of canal water over decades has induced a continual rise in the water table. In some areas the water table is already quite near the surface, thereby creating problems of waterlogging and salinity. Such conditions reduce the productivity of the land, or even take it out of production. On the other hand, too many tubewells in a region may result in a mining of the aquifer, and a possible influx of saline water into a sweet water aquifer.

Even under the assumption that ground and surface water can be properly managed so as to avoid disastrous future consequences, there is still the basic problem of water allocation. Water forms the lifeblood of agriculture in Pakistan, and is a scarce resource. As the flow of water can be controlled and diverted at many points in the surface water system, it is necessary to devise efficient water allocation schemes that will optimize some measure of regional welfare. This cannot be done without considering the use of water on individual farms. The Indus Basin Model Family was designed to relate agricultural development to the combined use and management of ground and surface water by incorporating these individual components into a unified mathematical framework.

The basic structure of the Indus Basin Model can be visualized as follows.
The entire basin is partitioned into 53 irrigated regions, referred to as polygons. Each polygon is essentially homogeneous with respect to ground water quality, and preserves boundaries that are significant to the ground water aquifer system. Linkages in water supply that arise from seepage of surface water to the aquifer and withdrawal of ground water via tubewells or capillary action are explicitly modeled for each polygon, thereby interlocking the polygons. Each polygon also receives surface water on a monthly basis from one or more control points of the surface water delivery system.

In addition to the above-mentioned water constraints, each polygonal model has embedded in it a single farm level model to characterize the agricultural production system of the area. Such a farm level model simulates the resource allocation choices of a single representative farmer which determine the production and disposition of 11 crops and 4 livestock commodities. Exogenous resource limitations are imposed on land, labor and canal water. The water supply and demand constraint of each farm level model includes estimates of water available from rainfall, evapotranspiration from the aquifer, the exogenous supply of canal and government tubewell water, and the endogenous supply of private tubewell water. However, when used to evaluate water allocation policies, canal water allocations are sometimes endogenous. There is, of course, great uncertainty associated with the availability of surface water, and estimates are based on long-term averages and worst cases. The uncertainty associated with the amount of rainfall is not so important as its contribution to the overall water supply is almost negligible.

Given the nature of the labor and capital markets in Pakistan, it is assumed that there are no labor or capital links between polygons. This implies that the only links between the polygons are the underground flows between them. Although polygons can be aggregated into either provinces or agro-climatic zones for reporting purposes, the lack of economic linkages between polygons eliminates the need to consider any intermediate level in the hierarchy between government and polygonal farmers. Another important simplification is the assumption of constant prices. This simplification can be justified in the case of major food grains and cash crops on the grounds that these prices are jointly determined by world prices and policy intervention. Livestock products and minor food grains are more of a problem. Given the size of the overall model, endogenizing prices is outside the scope of available software.

The entire Indus Basin Model is a linear programming problem with more than 20,000 constraints, thereby exceeding the capability of existing software for linear programs. It was apparent early on that a special simplification would be needed. By converting the height of the water table in each polygon to a policy instrument, structural simplifications could be made such that the entire model contains less than 8,000 constraints which is solvable using a large machine and commercial software.

The above introduction has been brief and has ignored many fine modeling
details. It also has made no mention of the extensive data gathering efforts that have taken place partially in support of the model. Despite this brevity and the lack of any mathematical equations (a detailed model statement is almost 40 pages), the reader must have gotten some impression of the size, structure and complexities that are captured by the system. With such a model, it is possible, among other things, to evaluate the effect of a wide range of water-related investment projects starting with dams and going all the way down to improvements on water courses in the field.

As individual farmers do not recognize their individual effects on groundwater equilibrium which must be maintained over the long-run, the government must take into account the long-term consequences of any water allocation scheme and the impact of water related investments on equilibrium. This expresses precisely the two-level aspect of the Indus Basin Model where some constraints are not formally recognized by the farmers (the policy receivers) even though the government (the policy maker) requires that they be satisfied. How the government might accomplish this task is explained in Section 4. First, however, we would like to introduce some basic notions on multi-level programming as most readers may not be familiar with this topic.

3. An introduction to multi-level programming

Whenever there is a set of nested optimization problems, one can speak of multi-level programming problems. This terminology was introduced by Candler and Norton [2], and several economic policy applications are discussed in their paper. A general mathematical definition of multi-level programming is not attempted in this paper as it involves an unwieldy expression of symbols and brackets. We have chosen to give a general definition of the two-level programming problem, leaving any generalizations to the reader. Assume that there is one policy maker (or ‘outer’ decision maker) and one group of independent policy receivers (the ‘inner’ decision maker). Let \( x \) be the \((n_1 \times 1)\)-vector of variables controlled by the outer decision maker, while the \((n_2 \times 1)\)-vector \( y \) contains the decision variables of the inner decision maker. Then the general two-level programming problem can be written as follows.

\[
\begin{align*}
\text{Min} & \quad f_2(x, y), \\
\text{s.t.} & \quad g_2(x, y) = 0, \quad h_2(x, y) \geq 0.
\end{align*}
\]  

(3.1)

\[
\begin{align*}
& \left\{ \text{Min} \quad f_1(x, y), \\
& \text{s.t.} \quad g_1(x, y) = 0, \quad h_1(x, y) \geq 0 \right\}.
\end{align*}
\]

Note that the inner optimization problem is nothing else but one of the constraints of the outer problem. In the inner optimization problem itself the \( x \)
variables are considered as given. The outer objective function could be vacuous, but if the inner one is not, one is still faced with a two-level programming problem. The above problem can be viewed as a two-player Stackelberg game employing the leader–follower solution concept. The leader is assumed to have complete knowledge of the behavior of the follower. It is precisely this authority between the leader and the follower that induces a natural solution to the above mathematical problem, provided, of course, that the optimal reaction \( y^*(x) \) is unique for the relevant choices of \( x \). If this is not the case, the above problem does not have a solution, and additional information must be added to the problem. Whenever \( y^*(x) \) is not convex, the overall problem has local optimal solutions. As one cannot assume a priori that \( y^*(x) \) is convex (even in the case of linear problems!), one must assume in general that the two-level programming problem is non-convex. This observation provides challenges to the mathematical programming community, but despair to all practitioners faced with multi-level programming problems and desiring globally optimal solutions. As we shall see next, however, there is a large class of linear two-level programming problems that can be solved for global solutions using mixed integer-linear programming.

Using the same \((x, y)\)-notation as before, consider the following two-level programming problem.

\[
\begin{align*}
\text{Min} & \quad c^1 x, \\
\text{s.t.} & \quad A_{21} y + A_{22} x \geq b_2, \quad x \geq 0, \\
& \{\min c^1 y + \frac{1}{2} y^T Q y + x^T C^1 y, \\
& \text{s.t.} \quad A_1 y + A_{12} x \geq b_1, \quad y \geq 0\}.
\end{align*}
\]

Here \( c^1 \) is \((1 \times n_1)\), \( Q \) is \((n_1 \times n_1)\) and symmetric, \( C^1 \) is \((n_2 \times n_1)\), and \( A_{ij} \) is \((m_i \times n_i)\) for \( i = 1, 2 \) and \( j = 1, 2 \). This is, at least to our knowledge, the largest class of two-level programming problems that is still solvable using commercial software. Consider the following rough derivation. If one writes the Kuhn-Tucker conditions of the inner optimization problem with \( x \) constant, one obtains a linear complementarity problem [3]. Embedding this linear complementarity problem in the outer optimization problem gives us problem (3.3).

\[
\begin{align*}
\text{Min} & \quad c^1 x, \\
\text{s.t.} & \quad (i) \begin{bmatrix} 0 & A_{11} & A_{12} \\ -A_{11}^T & Q & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \\
& \quad (ii) s, x, y, \lambda, \mu, \omega \geq 0, \\
& \quad (iii) \lambda^T s_i = 0, \quad i = 1, 2, \ldots, m_1, \\
& \quad \mu^T y_i = 0, \quad j = 1, 2, \ldots, n_1.
\end{align*}
\]
In the above augmented problem, only the constraints under (iii) are nonlinear. They can be replaced with so-called 'special ordered sets of type 1', where at most one of the variables in the set can be positive. Replacing the constraints under (iii) with \( m_1 + n_1 \) special ordered sets of this type, a mixed integer linear programming code such as APEX of CDC (which accepts continuous variables as part of its special ordered sets) can be used to solve the above problem [1].

The proposition of using a mixed integer linear programming code to solve linear two-level programming problems is an expensive one, as these codes are usually slow in finding a proven global optimal solution. It is certainly an unrealistic suggestion in the case of the Indus Basin Model where the augmented problem becomes too large for the machine. As it turns out, the Indus Basin Model contains specific characteristics that allows one to use linear programming techniques to solve the two-level programming problem.

4. The Indus Basin Model as a special case of multi-level programming

In the Indus Basin Model each polygonal representative farm is an independent unit, so that summing the individual farm-level objective functions over all 53 polygons provides us with a single objective function representing the entire agricultural sector. Maximizing the aggregated net farm income can be considered as a proxy for maximizing welfare. In that sense both the government and the farmers share the same objectives, and no special distinction needs to be made. The two groups do differ, however, in that the government wants to satisfy a set of political constraints and long-term ground water balance requirements that are outside the realm of the farmers, and, as such, are not recognized by them. The Indus Basin Model can therefore be viewed as a special example of a two-level programming problem where the outer objective function is vacuous. Using a notation similar to that of problem (3.2), we can write the following mathematical statement.

\[
\begin{align*}
\text{Minimize} & \quad c^1 y + d^1 y, \\
\text{s.t.} & \quad A_{11} y + A_{12} x = b_1, \quad y \geq 0.
\end{align*}
\] (4.1)

Here both \( x \) and \( d \) are considered policy variables that pertain to the government. The vector \( x \) contains all variables that are not under the direct control of the farmers, and that are not in \( d \). Surface water allocations to the farmers, for instance, comprise some of the \( x \)-components. The vector \( d \) represents a set of subsidies and taxes that the government wants to impose on farmers' water related activities. Note that these control variables enter the inner objective function in a multiplicative fashion, thereby making the above problem
bilinear. A setting of both $x$ and $d$ will result in a response $y^*$ by the farmers which in conjunction with the $x$ values may or may not satisfy the outer constraints of (4.1).

In order to solve the above problem (4.1), the following algorithm can be applied. Add for the moment the outer constraints to the inner optimization problem, and set the vector $d$ equal to zero. Next solve the resulting linear program, and denote the optimal solution as $(\tilde{x}, \tilde{y})$. Let the vector $\tilde{w}$ be the corresponding set of optimal dual prices associated with the outer constraints. Then if the government sets $x = \tilde{x}$ and $d^T = \tilde{d}^T = -\tilde{w}^T A_{21}$, one can verify, using duality theory, that $y^* = y$ is the optimal response for the farmers in the problem

\[
\begin{align*}
\text{Minimize} & \quad (c_1^T + d_1^T) y, \\
\text{s.t.} & \quad A_{11} y + A_{12} \tilde{x} = b_1, \quad y \geq 0. \tag{4.2}
\end{align*}
\]

By construction, the resulting pair $(\tilde{x}, y^*) = (\tilde{x}, \tilde{y})$ satisfies the outer constraints. The intuitive reason as to why $y^* = \tilde{y}$ solves problem (4.2) is that the transformation of the inner objective function from $c_1^T y$ to $(c_1^T + d_1^T) y$ has caused the outer constraints to become redundant as far as the farmers are concerned. Another way to express the same intuitive idea is to say that if the outer constraints were added to problem (4.2), their shadow prices would be zero.

It is important to note that after one computes the values of $(\tilde{x}, \tilde{d}, \tilde{y})$ for the augmented linear program, one must also solve the inner problem (4.2) with the policy variables fixed at $(\tilde{x}, \tilde{d})$. This is an unfortunate but necessary step to make sure that the optimal response $y^*(\tilde{x}, \tilde{d})$ is unique. As we noted in Section 3, without this uniqueness property, the solution to (4.1) is not defined. In this case the inner agent has no a priori reason to choose the value $\tilde{y}$ that also satisfies the outer constraints. A transfer payment (bribe) from the outer agent is needed to induce the inner agent to select the $\tilde{y}$ that satisfies the outer constraints.

Even though the vector $(\tilde{x}, \tilde{y}, \tilde{d})$ obtained by the above procedure solves the two-level programming problem (4.1), the actual values of $\tilde{x}$ and $\tilde{d}$ may not be politically acceptable. An example could be the surface water allocation scheme. As each representative polygonal farmer has an equal weight in the aggregated farmers' objective function, allocations of scarce water will naturally favor the efficient farmers. This could result in optimal water allocations $\tilde{x}$ that essentially ignore certain regions within the country. Such water allocations will be unacceptable to the regional government representatives who will insist that a minimum percentage of surface water must be allocated to them. This implies an increase in the number of outer constraints, which in turn requires the generation of a new solution $(\tilde{x}, \tilde{y}, \tilde{d})$. By comparing the optimal solutions and the corresponding objective function values, one can evaluate the impact of such political constraints.

Assuming that enough surface water allocation constraints have been added to the outer problem so as to render politically acceptable $x$-values, the resulting
solution vector \( d \) in the two-level problem may contain excessive taxes and/or subsidies. One approach is to design a set of acceptable tax/subsidy programs in the form of the vectors \( d_1, d_2, \ldots, d_l \). Then one can employ parametric programming on the inner objective function, and introduce subsequentially the tax packages \( d_i \) up to some specific level \( \lambda_i d_i \), where \( \lambda_i \) are scalars. If these packages are well designed, they will tend to lead toward approximate solutions of the outer constraints. Any violation of these constraints can then be corrected using appropriate government investment programs in tubewells, drainage projects and/or surface water related projects.

5. Some computational results

The Indus Basin study has not been completed at the time of this writing, and detailed results cannot be placed in the public domain. Nevertheless, we would like to report on three experiments (scenarios) involving the two-level formulation of the Indus Basin problem. The results of these scenarios are compared to the 'base case', which assumes that surface water allocations are fixed on the basis of a 5-year historical average, and that no ground water levels are to be enforced.

In scenario A, it is assumed that surface water allocations are fixed on the basis of a 5-year historical average (just as in the base case), and that the 1977–1978 levels of ground water are to be maintained. In Scenario B, it is assumed that a significant portion of surface water is allocated on the basis of existing historical water rights, and that the remaining portion is allocated freely by the model, while still requiring that the 1977–1978 ground water levels be maintained. In scenario C, it is assumed that all surface water is allocated freely by the model, and that the 1977 income levels of all the polygons are to be guaranteed, together with the requirement that the 1977–1978 ground water levels are maintained.

As these scenarios allow for increased freedom in the allocation of surface water, one would expect increases in agricultural production, and reductions in the levels of taxes and subsidies on water. These expectations are supported by the results of the experiments. In scenario A, enforcing the ground water balance requires taxes up to 250 rupees ($25.00) per acre foot and subsidies up to 110 rupees ($11.00) per acre foot for some polygons. Agricultural value added in domestic prices is decreased relative to the base scenario by approximately 1.5 billion rupees, which corresponds to a drop of roughly 5%. In scenario B, enforcing the ground water balance requires taxes of up to 125 rupees per acre foot, and subsidies of up to 105 rupees, while the corresponding taxes and subsidies in scenario C are up to 60 and 40 rupees, respectively. For scenario B, the increase in agricultural value added relative to the base case is 2.8 billion rupees, which corresponds to an addition of roughly 10%, while the correspond-
ing increase for scenario C is 4.3 billion rupees, or roughly 15%. Although one cannot make any definite conclusions, these experiments do point at the relative importance of flexible surface water allocations if ground water levels are to be maintained over long periods of time.

6. Summary and conclusion

In this paper, we have attempted to provide the reader with some insights into the complexities of an ongoing modeling exercise in a strategic planning environment. The main emphasis, however, has been on the multi-level aspects of the problem, explaining the roles of both the government and the farmers. As multi-level programming is not a widely known area within the field of mathematical programming, we have included an introduction to this important applied modeling tool. Although linear programming techniques are usually not applicable to the non-convex multi-level programming problem, it is shown how the special case of the Indus Basin problem forms an exception.

In conclusion, we would like to stress the importance of multi-level programming aspects in any policy-related exercise involving a hierarchy of optimizing agents. The behavioral roles of each agent must be spelled out clearly for both conceptual and computational reasons. In addition, we would like to express the need for more algorithmic developments pertaining to multi-level programming problems, since they occur frequently in a strategic planning environment.

References


No. 225. George Psacharopoulos, “The Economics of Higher Education in Developing Countries,” *Comparative Education Review*


No. 230. Abdun Noor, “Managing Adult Literacy Training,” *Prospects*


Issues of the World Bank Reprint Series are available free of charge from the address on the bottom of the back cover.