Labor Supply and Targeting in Poverty Alleviation Programs

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The introduction of variable labor supply raises some fundamental issues in analyzing the targeting of poverty alleviation programs in developing countries. It forces a reconsideration of the standard objective function, which is based on income or expenditure and so makes no allowance for the effort made in earning that income. We show that alternative views on the appropriate valuation of effort have very different implications for commodity-based targeting rules. We also establish a benchmark for marginal effective tax rates (inclusive of benefit withdrawal) in income-tested schemes and show that indicator targeting rules may also have to be modified significantly when labor supply responses are recognized.

For many governments of developing countries, finer targeting of programs to alleviate poverty appears an attractive option in an era of greatly constrained expenditure budgets. It seems as though policymakers could achieve greater poverty reduction with fewer resources if only they would resort to the magic of targeting. But fine targeting is not without its costs. It is now appreciated that the administrative costs of ensuring that benefits from a program reach the target group can be high (see Besley and Kanbur 1993). One response is to target by subsidizing commodities largely consumed by the poor or on the basis of other observable indicators—such as age, gender, region, or crop group—that are correlated with deprivation. There is now a literature on how such indicators might be used for optimal targeting (see, for instance, Akerlof 1978, Atkinson 1992, Besley and Kanbur 1988, and Kanbur 1987).

An aspect of the costs of fine targeting that has not been as well appreciated in the development literature as it should be is the effect on incentives. Consider, for example, the moves in Sri Lanka to target the rice ration subsidy in the wake of the economic reforms of the late 1970s (see Anand and Kanbur 1991). The system was transformed from one with a universal benefit to one in which the benefit was restricted (in principle) to those with incomes below a critical level.

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(although in practice, as discussed by Sahn and Alderman 1992, enforcement of
the income test was highly imperfect). Such income testing targets benefits on
the poor, but it is administratively costly and, in particular, has incentive effects
that diminish its appeal. Consider the position of a household that increases its
income, through its own efforts, to such an extent that it crosses the critical
threshold and so (with perfect enforcement) loses eligibility for the benefit. The
benefit's relation to income clearly reduces the incentive for the household to
increase its income. Such effects are a familiar part of the policy debate in
industrial countries (a representative example being Dilnot and Stark 1989), but
debate over such matters has not been prominent in developing countries. The
potential practical significance of incentive effects is emphasized, however, by
the empirical work of Sahn and Alderman (1992). They find that the Sri Lankan
rice subsidy was associated with a substantial reduction in the labor supply of
recipients, with potentially important implications for the evaluation of the
scheme.

The purpose of this article is therefore to take up the broad issue of targeting
and incentives in developing countries and, in particular, to explore the implica-
tions of variable labor supply for the design of poverty alleviation programs. It
will be seen that once the potential incentive effects of such programs are recog-
nized, previous discussions of optimal targeting require revision. Simple rules of
thumb—for example, “spend more on the group with greater measured
poverty”—have to be modified to take into account such features as differing
labor supply elasticities. Similarly, commodity-based targeting rules need to be
modified.

The general issues have been extensively addressed in the optimal tax litera-
ture, with tools and results that can be borrowed for the analysis of the incentive
effects of targeting in developing countries. The approach pursued, however,
departs from the usual optimal tax tradition in two ways. First, we take the
objective of policy to be the minimization of a poverty index rather than the
maximization of a social welfare function. This approach is not without its
critics (see, for instance, Stern 1987), but in a technical sense at least it is
relatively straightforward. Loosely speaking, a poverty index defined on
utility—attaching zero weight to all households above some threshold—is
merely a special form of social welfare function.

Our second point of departure is more fundamental. The motivation for this
departure begins with the observation that the poverty indexes on which much
policy discussion focuses are, in practice, almost invariably defined in terms not
of utility but of income. In the presence of incentive effects, these criteria are
very different things (and can move in different directions). Indexes that focus
on income attach no significance to the effort put into earning income or, put
another way, attach no weight to the leisure of the poor. And indeed it is clear
that much policy debate is cast in precisely these terms: the focus is on the
income of the poor, not on how hard they work to get it. This is not to deny
that, from the Factory Acts to the Maastricht Treaty, policymakers have been
concerned about avoiding excessive work hours. In 1904, for instance, Churchill was arguing eloquently that “[working people] demand time to look about them, time to see their homes by daylight, time to see their children, time to think and read and cultivate their gardens—time, in short, to live” (quoted in Gilbert 1991: 196). The point is rather one of emphasis.

A large part of our purpose here is to explore the implications of a nonwelfarist approach to policy analysis. We do not necessarily advocate the evaluation of policy by its effects on some income-based poverty index, but it may nevertheless be useful to explore the implications of doing so, for two reasons. First, to the extent that—right or wrong—policy is often evaluated, at least in part, by the use of such indexes, it is helpful to know what kind of policy would be implied by the explicit pursuit of such a minimand. Second, following the work of Sen (1985), there has been growing interest in nonwelfarist approaches to policy analysis. Yet there has been relatively little formal work along these lines (an exception is Ulph 1991). The work reported here may contribute to the development of this research agenda. It should immediately be stressed, however, that we do not set out to capture the richness of Sen’s capabilities approach. For although there is only one kind of welfarism, there are potentially very many kinds of nonwelfarism. Here we report on and extend recent work on one perhaps crude form of nonwelfarism, one that has the merits of capturing the common preoccupation with income-based measures of poverty and, moreover, of being readily tractable.

Section I tackles the underlying conceptual problem: the measurement of poverty when labor supply is variable. The rest of the article applies this nonwelfarist approach to three aspects of targeting, focusing in each case on the role played by labor supply responses. Section II considers commodity-based targeting, section III analyzes income-based targeting, and section IV discusses the use of other observable characteristics for indicator targeting. Section V offers conclusions.

I. Labor Supply and the Measurement of Poverty

The measurement of poverty is of course a vast topic in itself, and we make no attempt to survey it here (see Atkinson 1987 and, for a general survey, Ravallion 1993). But before addressing the implications of variable labor supply for the design of poverty alleviation programs, we need to specify the way in which labor supply affects the perceived extent of poverty, this being the presumed minimand for the policy exercises in subsequent sections. The first task is thus to consider the ways in which variable labor supply might be incorporated into the measurement of poverty. These issues have been little discussed and, as will be seen, remain somewhat perplexing.

The standard approach to the measurement of poverty proceeds by comparing \( y^b \), the income some individual \( h \) has available to spend, to a poverty
Because \( y^h \leq z \), aggregate poverty might be measured by an index of the form

\[
P = \int_0^\infty D(z, y) f(y) \, dy
\]

where \( D(z, y) \) can be thought of as the deprivation of an individual with income \( y \), and \( f(y) \) denotes the density of \( y \). This is a fairly general form of poverty index, encompassing a range of widely used measures as special cases. (The only substantive restriction is additive separability, which precludes Gini-based indexes such as that of Sen 1976.) If \( D(z, y) = \max (z - y, 0) \), for example, then \( P \) is the aggregate poverty gap. For our purposes, we need not assume any particular form for \( D(\cdot) \) beyond making the natural assumptions that deprivation is positive only for the poor, so that \( D(z, y) \) is strictly positive if \( y < z \) and zero otherwise, and decreases as income rises toward the poverty line (so that \( D_y < 0 \) for \( y < z \), with the subscript indicating differentiation).

In the absence of labor supply responses, and assuming there to be only a single consumption good, there are two very different ways in which this approach to measuring poverty might be justified. The first is welfarist in the sense that the primitive concern in identifying and quantifying poverty is with individuals' realized levels of welfare. In this view, a household is poor if and only if it fails to achieve some poverty line utility level, \( u_z \). With only a single consumption good, \( h \)'s utility is simply \( u(x^h) \), where \( u(\cdot) \) is the direct utility function and \( x \) denotes consumption of the single good. Because \( x^h = y^h \) from \( h \)'s budget constraint, the condition \( u(x^h) \leq u_z \) is equivalent to \( y^h \leq z \), where \( z = u^{-1}(u_z) \). The poverty index \( P \) simply puts a metric on the shortfall of utilities from \( u_z \).

The second justification of equation 1 makes no appeal to notions of utility and is in that sense nonwelfarist. It views poverty as the inability to acquire an amount \( z \) of the consumption good. The primitive concern is with the potential to consume rather than the well-being derived from doing so.

In the simplest case—which is the case implicitly assumed in much of the literature—the welfarist and nonwelfarist approaches are thus indistinguishable in terms of poverty measurement. The equivalence collapses, however, when labor supply responses are admitted, because the assumption of one good is relaxed. Households acquire the consumption good, at least in part, by forgoing another good, leisure. Some way must then be found to compare deprivation across alternative bundles of consumption \( x \) (now thought of as an \( N \)-vector) and labor supply \( L \). The welfarist will use individuals' own preferences to make the comparison; a nonwelfarist may not.

Note that, in a sense, there is nothing special about labor supply here. The same issue—that of deciding how to evaluate the deprivation associated with distinct bundles—would arise if labor supply were fixed but there were two consumption goods. Rather the point is that labor supply makes the issue.

1. The terms "individual" and "household" are used synonymously in what follows, the issues raised by the distinction between the two being somewhat removed from the central concerns here.
unavoidable and has a distinctive feature that adds a further layer of complexity. Although it is usually reasonable to assume that consumer and producer prices for consumption goods do not vary across households, wage rates clearly do. Indeed, the tradition of the literature on optimal income tax is to view variation in the wage rate—ability, broadly interpreted—as essentially the only way in which households differ.

To bring out these points concretely, consider first a welfarist approach to the measurement of poverty in the presence of labor supply responses. Starting with a poverty line utility level $u_z$, define the indirect utility function $V(q, w, B)$, giving the maximum utility that can be attained at consumer prices $q = (q_i)$ for the $N$ goods at wage rate $w$ and with lump-sum income $B$. We define the lump-sum income to be exclusive of the value of the individual's endowment of time and, perhaps, goods. Assuming that consumer prices are common across households, the utility achieved by $h$ is thus $V(q, w^h, B^h)$; so $h$ is poor if

$$V(q, w^h, B^h) \leq u_z.$$

To move from utility to income space, follow King (1983) in defining the equivalent income function $y_E(q, q^R, w^h, w^R, B^h)$ by

$$V(q^R, w^R, y_E) = V(q, w^h, B^h).$$

That is, $y_E$ is the lump-sum income at which $h$ would be as well off when facing reference consumer prices $q^R$ and a reference wage $w^R$ as in the situation being evaluated (the latter being described by the consumer prices $q$, wage rate $w^h$, and lump-sum income $B^h$ that $h$ actually faces). The condition (equation 2) for $h$ to be poor is then equivalent to $y_E \leq z$, where $z$ is now defined by $V(q^R, w^R, z) = u_z$, and poverty is naturally measured by

$$P = \int_{0}^{\tilde{w}} D(z, y_E) g(w) \, dw$$

where $\tilde{w}$ is the poverty wage defined by $z = y_E(q, q^R, \tilde{w}, w^R, 0)$, and $g(w)$ is the density of wages. (To avoid a double integral we assume, here and henceforth, that $B^h = 0, \forall h$.) In principle, there is thus no difficulty in developing welfarist measures of poverty in the presence of labor supply responses and household-specific wage rates. Note, though, the element of arbitrariness in the choice of reference prices and, in particular, the reference wage for the evaluation of equivalent incomes. Different choices of reference prices may lead to different rankings of poverty alleviation strategies.

An alternative to developing a welfarist measure of poverty would be to generalize the nonwelfarist approach. As noted in the introduction, there are

2. Use of the equivalent income function is not unproblematic: Blackorby and Donaldson (1988) show that it is not in general concave (in the underlying consumption bundle). In the present context, this is liable to mean, for instance, that transferring commodities from a poor person to a richer person could actually reduce the aggregate poverty gap measured in terms of $y_E$.

3. To be precise, what really matters is the vector of relative reference prices, $q^R/w^R$. Because indirect utility is homogeneous of degree zero in prices and income, it follows from equation 3 that equivalent income measured in units of leisure (that is, $y_E/w^R$) depends on reference prices only through $q^R/w^R$. 
many conceivable kinds of nonwelfarism. For example, one might focus on the income that could be earned by working an acceptable number of hours. The implications of this approach for the empirical measurement of poverty have indeed been explored by Garfinkel and Haveman (1977) and by Haveman and Buron (1993). It may well be that their approach—perhaps closer to the capabilities notion—yields qualitatively very different conclusions from that pursued here. Kanbur and Keen (1989) show that the two approaches—of using what they call standard and received income—do indeed have distinct implications for the design of a linear income tax. We pursue only one kind of nonwelfarism here, not because we advocate it, but for brevity and because it seems to capture much of the common tone of policy discussion.

In generalizing the nonwelfarist approach, one starting point is the specification of a particular bundle of consumption and labor supply to act as the reference for evaluating actual bundles. This target \((N + 1) - \text{vector} (x^*, L^*)\) is generated not by any reference to utility but by prior views as to what consumption standards households need to attain. To avert deprivation, for instance, households ought to be able to attain a reasonable nutritional intake without an excessive amount of work effort. The question then becomes how to measure the distance between an individual's actual consumption vector \((x, L)\) and the target vector. There are an infinite number of possible metrics.

For simplicity, we consider here only deprivation measures of the form \(D[z, y(q, w)]\), where

\[
z^h = s_x^h \cdot x^* - s_L^h L^*
\]

and

\[
y(q, w^h) = s_x \cdot x(q, w^h) - s_L L(q, w^h)
\]

where \(x(\cdot)\) and \(L(\cdot)\) denote the Marshallian commodity demand and labor supply functions. In equation 5, \(z^h\) is a poverty line defined as the value, at some shadow prices \(S^h = (s_x^h, s_L^h) > 0\), of the resources needed to attain the target vector. In equation 6, \(y(q, w^h)\) is the shadow value of the net resources actually enjoyed by \(h\). Assuming that \(y\) is strictly increasing in \(w\) and that \(z^h\) does not increase too rapidly with \(w^h\) (typically it would fall), there exists a unique poverty line wage \(w^*\) at which \(y(q, w^*) = z\), and poverty can be measured as

\[
P = \int_0^{w^*} D[z, y(q, w)] g(w) \, dw.
\]

Consider, for concreteness, the case in which \(D(\cdot)\) depends on the poverty gap \(z - y(w)\). Deprivation, and hence poverty, can then be assessed by asking, by how much do the resources that \(h\) needs to attain the target vector \((x^*, L^*)\) exceed those actually made available to \(h\)?

The question, then, is how to value these resources—the specification, that is, of the shadow prices \(S^h\). There are an infinite number of possible choices, and few natural axioms to invoke. Both technological and ethical considerations arise. Emphasizing the former, a natural approach is to value resources at pro-
ducer prices. This can be done by taking $s_x = p = q - t$, where $p$ denotes producer prices, $t$ denotes the vector of commodity taxes, and (assuming that earned income is untaxed) $s_p^L = w^b$. The value judgments underlying the non-welfarist approach, however, may point toward other shadow prices. It may be, for instance, that some goods are felt to be irrelevant to the achievement of minimal living standards, in which case their appropriate shadow price is zero.

Of more particular importance to our concerns here, policymakers often seem to attach positive weight to the capacity to consume but zero weight to the enjoyment of leisure. Put crudely, policymakers may not care how hard people have to work so long as they are able to sustain a decent level of consumption. Taken to the extreme, such a view corresponds to $s_L^L = 0$. An alternative and very different ethical position is that work effort in excess of the target $L^*$ — or, underenjoyment of leisure in relation to some target — should not only be valued positively, but, at a minimum for the least able, should be valued at a shadow wage in excess of the actual wage $w^b$. Taking $s_L^L = w^b$ implies that a given shortfall in leisure hours in relation to the target translates into less deprivation for a low-paid individual than for a high-paid one.

These issues will not be resolved here, but we focus on their implications for the incorporation of labor supply responses into the analysis of poverty alleviation. We simply assume, when defining poverty as in equation 7, that $s_x = p$. That is, we assume that consumption goods are valued at producer prices but for the moment the value placed on leisure, $s_L^L$, is unrestricted. We assume, however, that $s_L^L$ is independent of consumer prices, $q$. This seems a reasonable simplification for our purposes, there being no instantly compelling reason to suppose that the extent of deprivation in excessively hard work depends on the prices of the goods it buys.

II. TARGETING BY COMMODITIES

Which kinds of goods should be subsidized by a government seeking to alleviate poverty, and which should be taxed? Specifically, suppose there are two commodities, 1 and 2, that the government can subsidize or tax as it pleases subject only to an overall budget constraint. Labor supply is variable — the implications of this being our central concern — but labor income cannot be taxed. Starting from a position in which neither good is subsidized (or taxed), what is the effect on aggregate poverty, $P$, of introducing a small subsidy on good 2 financed by a tax on good 1?

Besley and Kanbur (1988) address this question in a welfarist context. They effectively assume labor supply responses to be zero, but it is straightforward to show that relaxing this assumption does not affect their central result. Defining poverty in terms of equivalent income (taking the reference wage for each household to be its actual wage), the effect on $P$ of introducing a small tax on good 1 (and hence a small subsidy on good 2) can be shown to be
Recalling that $D_y < 0$, the interpretation of equation 8 is straightforward: good 2 should be subsidized by a tax on good 1 if and only if consumption of good 2 is more heavily concentrated among the poor than is consumption of good 1 (weighted, by the terms $D_y$, to attach most importance to the most deprived households). Note that the convenient absence of price elasticities reflects the assumption that the starting point is one with no taxes or subsidies. When considering large reforms, considerations of excess burden will also arise, bringing into play elasticities of demand for consumer goods and hence labor supply responses.

The condition expressed in equation 8 is essentially the same form as the result of Besley and Kanbur (1988, equation 26). The only difference is that Besley and Kanbur work with the poverty index of Foster, Greer, and Thorbecke (1984) rather than with the more general form used here. Variable labor supply thus makes no difference to the welfarist analysis. The impact on a household's welfare of a change in the consumer price of good $i$ is simply proportional to its consumption of $i$. The induced effect on the pattern of consumption—and, by the same token, on labor supply—drops out by the envelope property. That is, because the initial level of labor supply is chosen by the individual so as to maximize utility, a small change in that level—as might be induced by the commodity price change associated with the tax/subsidy scheme—will have no (first-order) effect on welfare and hence also no effect on poverty defined in welfarist terms.

Consider now the same problem from the nonwelfarist perspective. It is shown in the appendix that the effect on a poverty index of the form in equation 7 of the revenue-neutral introduction of a small subsidy on good 2 is given by

$$\frac{\partial P}{\partial t_1} = (\bar{x}_i) \int_0^{\bar{u}} (A_2 - A_1) D_y [z^*, y(q, w)] g(w) dw$$

where

$$A_i = \bar{x}_i + (s^i - w^*) \frac{\partial L/\partial q_i}{\bar{x}_i}. \tag{10}$$

Comparing equations 9 and 10 with equation 8, the sole consequence of adopting the nonwelfarist approach\(^5\) to the targeting of subsidies is thus to introduce labor supply considerations in the form of the terms $(s^i - w^*) \frac{\partial L/\partial q_i}{\bar{x}_i}$.

4. In equation 8, $\bar{u}$ is a poverty line wage defined by $\gamma(q, q^*, \bar{w}, \bar{u}) = z$. For brevity, the derivation of equation 8 is omitted. It is similar to that of equation 9, which is sketched in the appendix. The critical step is to note (in place of equations A-5 and A-6) that $\partial y/\partial q_i = -x_i$ at $q^* = q$ and $w^* = w^k$. The simplicity of equation 8 would not be obtained if the reference wage were specified to be other than $w^k$.

5. Here and elsewhere, "the" nonwelfarist approach refers to the particular variant of nonwelfarism described in the previous section.
There are two cases of interest in which these terms vanish. The first, trivially, is that in which labor supply decisions are unaffected by the consumer prices of the goods being studied \((\partial L/\partial q_i = 0)\). In practice, it may often be tempting to assume that this is indeed the case. As an antidote to routinely doing so, however, it is worth recalling that (in the absence of lump-sum income) to assume labor supply to be independent of all consumer prices is to assume it also to be independent of the wage rate. This follows from homogeneity of degree zero of the labor supply function \(L(q, w)\). The second case in which these labor supply terms disappear is that in which the leisure component of an individual's deprivation is valued at the individual's wage rate \((s^L = w^h)\). The intuition for this is that with \(s^L = q\) and \(s^L = w^h\), the impact on deprivation of behavioral responses to the tax reform is being evaluated at the prices actually faced by the consumer; just as those responses can have no effect on the consumer's budget constraint, so too they can have no effect on measured deprivation.

But there are, of course, other possible choices of \(s^L\). One is to attach no weight at all to labor supply \((s^L = 0)\). Then equation 10 becomes

\[
A_i = \frac{b_i - \epsilon_{Li}}{b_i}
\]

where \(b_i = q x_i / w L\) denotes the budget share of good \(i\), \(b_i\) its mean, and \(\epsilon_{Li} = (\partial \ln L / \partial \ln q_i)\) the elasticity of labor supply with respect to \(q_i\). Other things being equal, the case for subsidizing good \(i\) is thus weaker the more positive \(\epsilon_{Li}\) is, that is, the more such a subsidy would tend to reduce labor supply. This is reminiscent of optimal tax arguments and points to relatively heavy taxation of relatively strong complements with leisure—but the underlying reasoning is very different. With \(s^L = 0\), the sole object of policy is to push poor individuals' consumption as far toward the target consumption vector \((x^*_1, x^*_2)\) as possible. There are, broadly, two ways of doing this. The first is to deploy subsidies that enable households to afford greater quantities at any given income. The other is to encourage households to increase their earnings and so to purchase more at any given price. It is on this second front that complementarities between goods and leisure enter the picture. Expanding the consumption of (relative) complements with leisure may be more expensive than expanding the consumption of substitutes. The effects of subsidizing the former are liable to be at least partly offset by an induced reduction in disposable income, whereas subsidizing the latter generates a reinforcing expansion of income.

For reasons discussed above, however, it might be preferable to attach a large positive weight to leisure in measuring deprivation. The implications of doing so are not merely qualitative; they can reverse the conclusions of the previous

6. Some emphasis should be put on the word "relatively." It is not necessary for the arguments here that there exist any good \(j\) that is complementary with leisure in the sense that the compensated demand for \(j\) falls as \(w\) rises; indeed, there may exist no such good. It is the degree of complementarity that is important. It is convenient, for clarity, to speak of taxing or subsidizing complements with leisure. The more delicate and exact formulation of the argument is straightforward, but cumbersome.
paragraph. Suppose, for instance, that $s_P^p$ is at or above the poverty line wage $w^p$. Other things being equal, the goods that ideally should be subsidized to alleviate poverty are thus precisely those for which $\partial L/\partial q_i > 0$. Instincts honed on the optimal tax literature are confounded because complements with leisure should be subsidized rather than taxed. The reason, however, is straightforward. With $s_P^p > 0$, one way of reducing deprivation is by reducing the work effort of the poor. Less effort means less consumption and so a greater shortfall from $x^*$; but with $s_x^p$ equal to the prices faced by the consumer and $s_P^p > w^p$, the deprivation measure attaches more weight to the hour of leisure gained than it does to the consumption consequently forgone.

Consider, for instance, the appropriate treatment of food. In the developing-country context it could plausibly be argued that greater food intake might enhance the capacity to work. Food and leisure would then be thought of as substitutes: a reduction in the consumer price of food will tend to increase labor supply (so that $e_L,food < 0$). With $s_L^p = 0$, the implication of the analysis above is that (other things being equal) food should be subsidized, reflecting the secondary benefit of expanded income that such a subsidy induces. If, however, $s_P^p$ is above the poverty line wage, the implication is that food should be taxed. This may at first seem strange, but it has a straightforward explanation. Although taxing food in itself increases deprivation, this may be more than offset by the expanded consumption of both leisure and commodity 2 in the background, the latter now being subsidized by receipts from the food tax.

Pursuing the nonwelfarist approach, rules of thumb for commodity targeting are thus highly sensitive to the weight attached to deprivation of leisure. The issue is troublesome. Having raised it, however, we now put it aside by assuming in the next two sections that $s_L^p = 0$. Because there will also be only one consumption good—commodity subsidies not being at issue—the nonwelfarist deprivation measure in what follows is simply the shortfall of aggregate consumption (equivalently, of net income) from some poverty line.

### III. Targeting by Income

In the absence of incentive effects (and with sufficient resources available for poverty relief) the design of income-based targeting is a trivial exercise. After the poverty line is established, those individuals who are initially below it are given exactly that transfer needed to bring them just above it. $^7$ Such a scheme involves no leakages. If there are no labor supply or other effects in transferring or raising these resources, and if the informational and administrative requirements can be met without cost, this method gives perfect targeting. But once incentive effects are admitted, the difficulties noted in the introduction arise. Because perfect targeting implies an effective marginal tax rate of 100 percent on those below

$^7$ Bourguignon and Fields (1990) examine the optimal poverty alleviation strategy (in the absence of behavioral responses) when the available budget is insufficient to eliminate poverty.
the poverty line, the poor have no incentive to earn income. Their rational labor supply decisions would then be likely to greatly increase the revenue costs of alleviating their poverty. Incentive effects thus rule out marginal rates of 100 percent on the poor. The questions of precisely how high or low those rates should be, and of how they should vary with income, then become considerably more complex.

There is a large literature, initiated by Mirrlees (1971), that addresses the optimal design of nonlinear income taxes in a welfarist setting. In this work, the issue of incentives for supplying labor is tackled directly by modeling individuals as choosing between work and leisure given the tax-transfer schedule they face. There are assumed to be a large number of individuals, differing only in the pretax wage they can earn. (We relax this homogeneity assumption in the next section.) The government then chooses a schedule that maximizes a social welfare function defined on individuals' welfare, that is, on the utility they derive from their consumption-leisure bundles. As noted in the introduction, however, there is a striking and fundamental dissonance between this welfarist approach and the tone of much policy debate. It is the consequences of reform for the incomes of the poor—the money in their pockets, not something akin to money metric measures of their welfare—that are commonly discussed and analyzed. Kanbur, Keen, and Tuomala (forthcoming) therefore examine the implications of an alternative approach to the design of nonlinear income tax schemes. Besley and Coate (1992) adopt a similar approach in analyzing the case for workfare schemes. In Kanbur, Keen, and Tuomala's approach, the objective of policy is to minimize an income-based poverty index rather than to maximize social welfare. This section reviews their conclusions.

We begin by recalling the main lessons from the welfarist literature on optimal nonlinear income taxation (as reviewed, for instance, in Tuomala 1990). Three general qualitative conclusions emerge:

- The marginal tax rate should everywhere be non-negative.
- The marginal tax rate on the lowest earner should be zero so long as everyone supplies some labor at the optimum.
- The marginal tax rate on the highest earner should be zero so long as wages in the population are bounded above.

The first result is more striking than is commonly recognized. Although it may well be optimal for the average tax rate on the least well off to be negative, it cannot be desirable to subsidize their earnings at the margin. The limitations of the second and third results concerning the endpoints are well known: simulations suggest that zero may be a bad approximation to optimal marginal tax rates in the tails of the distribution, and it can be shown that if it is optimal for some not to work, then the optimal marginal tax rate at the bottom of the income distribution is strictly positive (Tuomala 1990). Nevertheless, these results continue to color professional thinking on issues of rate structure. The lower endpoint result, in particular, has been taken as suggestive in arguing...
against very high effective marginal rates on the poor (as, for instance, in Kay and King 1986).

Do these conclusions continue to apply when the objective of policy is not the maximization of social welfare but the minimization of income poverty? The third result certainly does (see Kanbur, Keen, and Tuomala forthcoming for proofs of this and the claims below). As expected, the third result applies because in the context of poverty alleviation the only reason to care about the highest earner—indeed about any of the nonpoor—is as a source of revenue. It is well known that in these circumstances the marginal tax rate on the highest earner should be zero; if it were strictly positive, additional revenue could be extracted by slightly lowering it and thereby inducing the highest earner to earn additional taxable income.

The first and second results, in contrast, are overturned if the objective is to minimize income poverty. If it is optimal for the lowest-ability earner to work, the marginal tax rate at the lower end of the distribution should be strictly negative; that is, a marginal earnings subsidy should be paid to the very poorest. To see why this is optimal from the nonwelfarist perspective even though it cannot possibly be optimal from a welfarist one, consider an initial position in which the individual with the lowest ability works and faces a strictly negative marginal tax rate. Imagine now increasing the marginal tax rate faced by this individual while leaving the average rate at the individual’s initial gross income unchanged. The effects of this rotation of the poorest worker’s budget constraint through the initial consumption-leisure bundle are that the individual’s welfare rises (because if the initial consumption-leisure bundle remains feasible, any change in the individual’s behavior must signify an increase in welfare); the individual’s net income falls (because the only incentive effect is a substitution toward leisure induced by the higher marginal tax rate); and the government’s revenue increases (because the subsidy is paid at a lower rate on a narrower base). From the welfarist perspective, the combination of the utility gain to the individual and the revenue gain to the government makes this reform unambiguously desirable. From the nonwelfarist perspective, however, opposing effects are at work. The revenue gain is desirable, but the net income loss to the poorest worker is not. Minimization of an income-based poverty index will require striking a balance between the two effects, which will make a marginal subsidy on the very poorest optimal.

The possibility of an optimally negative marginal tax rate is confined, however, to the poorest of the poor. For those who find themselves exactly at the poverty line, the optimal marginal rate can be shown to be strictly positive.

These qualitative implications of the nonwelfarist approach thus point to a pattern of marginal tax rates below the poverty line that is both complex and potentially very different from that suggested by the welfarist tradition. But how far do low or even negative marginal tax rates on the very poorest individuals extend into the range of incomes? And how is the poverty-minimizing rate
structure affected by the precise location of the poverty line \( z \) and by the form of the deprivation function \( D(\cdot) \)?

Table 1 reports results of simulations intended to address these concerns. The results assume Cobb-Douglas preferences

\[
u(x, L) = (1 - \delta) \ln(x) + \delta \ln(1 - L) \quad \delta \in (0, 1)
\]

(the time endowment being normalized at unity) with \( \delta = 1/2, \ln(w) \) normally distributed (with mean \(-1\) and standard deviation \(0.39\)), and that the revenue requirement is about 10 percent of gross income. These are the standard assumptions in simulations of this sort. The novelty is in the form of the objective function, for which we take a poverty index of the form developed by Foster, Greer, and Thorbecke (1984):

\[
P^a = \int_0^\infty \left( \frac{x(w) - z}{z} \right)^a g(w) \, dw \quad \alpha > 1
\]

<table>
<thead>
<tr>
<th>Poverty line and percentile of the wage distribution</th>
<th>Average tax rate</th>
<th>Marginal tax rate</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>a. Low poverty line</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low*</td>
<td>0.06</td>
<td>-100</td>
</tr>
<tr>
<td>Poverty line</td>
<td>0.31</td>
<td>-3</td>
</tr>
<tr>
<td>0.50</td>
<td>12</td>
<td>53</td>
</tr>
<tr>
<td>0.90</td>
<td>29</td>
<td>35</td>
</tr>
<tr>
<td>High</td>
<td>0.99</td>
<td>29</td>
</tr>
<tr>
<td><strong>b. Middle poverty line</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low*</td>
<td>0.02</td>
<td>-100</td>
</tr>
<tr>
<td>Poverty line</td>
<td>0.43</td>
<td>0</td>
</tr>
<tr>
<td>0.50</td>
<td>9</td>
<td>53</td>
</tr>
<tr>
<td>0.90</td>
<td>27</td>
<td>34</td>
</tr>
<tr>
<td>High</td>
<td>0.99</td>
<td>27</td>
</tr>
<tr>
<td><strong>c. High poverty line</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low*</td>
<td>0.003</td>
<td>-87</td>
</tr>
<tr>
<td>0.50</td>
<td>8</td>
<td>54</td>
</tr>
<tr>
<td>0.90</td>
<td>26</td>
<td>34</td>
</tr>
<tr>
<td>High</td>
<td>0.99</td>
<td>26</td>
</tr>
<tr>
<td><strong>d. Maximin</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low*</td>
<td>0.16</td>
<td>-100</td>
</tr>
<tr>
<td>0.50</td>
<td>17</td>
<td>53</td>
</tr>
<tr>
<td>0.90</td>
<td>32</td>
<td>35</td>
</tr>
<tr>
<td>High</td>
<td>0.99</td>
<td>32</td>
</tr>
</tbody>
</table>

Note: In all four groups, the ratio of aggregate consumption to aggregate output is 0.9. In groups a, b, and c, the parameter for aversion to inequality among the poor, \( \alpha \), is 2. In groups a, b, and c, the minimum level of consumption, \( x(n_0) \), is 0.06; in d it is 0.07.

a. The percentile of the wage distribution below which individuals choose not to work.

b. Assumes infinite aversion to inequality among the poor \( (\alpha = \infty) \).

Source: Authors' calculations.
where \( x(w) \) denotes the consumption of an individual with wage \( w \). The parameter \( \alpha \) in equation 13 provides a convenient parameterization of alternative degrees of aversion to inequality among the poor.

One immediate implication of this specification should be noted. With Cobb-Douglas preferences (so that the marginal rate of substitution between consumption and work is strictly positive at zero hours) and a lognormal wage distribution (so that the lower bound of \( w \) is zero), there are some who will work only if the marginal tax rate at the bottom of the distribution is infinitely negative. In both the welfarist context and that of income poverty minimization, it would be optimal to have some of the population idle. As noted above, in the welfarist case the optimal marginal rate at the bottom of the income distribution is then strictly positive. However, for the case in which the objective is to minimize income poverty and some households are idle at the optimum, we have been unable to derive any general result on the sign of the optimal marginal rate at the lower endpoint. The simulations provide some indication of the extent to which the argument for nonpositive marginal rates at the lower end (when the poorest work) continues to exert some force when instead the wage distribution is not bounded away from zero. Table 1 gives optimal average and marginal tax rates at various percentiles of the wage distribution, starting at the bottom and including the point at which the assumed poverty line is to be found. The first three groups (a, b, and c) all take \( \alpha = 2 \) and differ in taking successively higher poverty lines. The last (d) looks at the maximin case, which corresponds to \( \alpha = \infty \).

Several features stand out in the table. First, the marginal rate on the lowest gross income—which, as just noted, we are unable to sign in principle—emerges as very strongly positive: not only is it not negative, it is not even low. Second, marginal tax rates decline monotonically from the poorest to the richest individual, implying that the dictates of effective targeting can run exactly counter to the popular notion that equity concerns require the marginal tax rate to increase with income. Such declining marginal tax rates run counter to the conclusion sometimes drawn from the welfarist literature that the administrative advantages of linear taxation can be bought at relatively little loss in terms of policy effectiveness.

The third feature of the table is that (comparing a and c) increases in the poverty line reduce optimal marginal rates at and below the poverty line. The intuition for this seems to be that the case for low marginal rates intended to encourage those at or near the poverty line to move out of poverty, becomes stronger as the poverty line moves into denser parts of the distribution. Fourth, comparing the maximin case with the others, increases in the extent of aversion to inequality among the poor tend to increase the marginal rates that they optimally face. Other simulations (not reported here) suggest that moderate variation in the revenue requirement affects the general level of marginal tax rates (which tend to increase with the revenue required) but not the qualitative pattern of their variation with income. This is perhaps as would be expected.
because the greater the concern with alleviating poverty, the more attractive schemes that approach minimum income guarantees are likely to be: the emphasis is then on raising the consumption of the very poorest, and financing the transfer this requires calls for relatively high marginal tax rates in the lower part of the distribution in order to impose sufficiently high average tax rates further up the distribution.

But perhaps the most important feature of the results is the finding of marginal tax rates on the poor that are invariably rather high (bearing in mind the fairly minimal revenue requirement). In most cases marginal rates on the bulk of the poor exceed 60 percent, and in all cases they exceed 50 percent. The case for low marginal tax rates to encourage the poor to help themselves thus is less discernible in the simulations than expected. Even with the relatively elastic labor supply responses implicit in Cobb-Douglas preferences (the elasticity of substitution between consumption and leisure being unity), a stronger mark is left by the case for high marginal rates associated with the unattainable ideal of perfect targeting described at the start of this section. Simulations for the case in which the elasticity of substitution is 0.5 (reported in Kanbur, Keen, and Tuomala forthcoming) confirm this impression.

The optimal marginal tax rates that emerge from these simulations are not necessarily higher in the nonwelfarist case than in the welfarist one. Indeed, it is not clear that a coherent comparison between the two approaches can be made because the latter, but not the former, depends on the cardinal representation of preferences. The safest conclusion—albeit a provisional one, because our simulations are inevitably only special cases—seems to be that a concern with income poverty does not in itself provide a strong case for marginal tax rates on the bulk of the poor that are substantially lower than expected from the perspective of the welfarist tradition. The reason for this, it seems, is that shifting from the welfarist to the nonwelfarist perspective introduces two considerations that point in opposite directions. First, the case for lower marginal tax rates on the poor is strengthened by the prospect of inducing them to raise their own incomes. The nonwelfarist view attaches no weight to the leisure that the poor forgo; this underlies the result that a marginal earnings subsidy on the very poorest is optimal when that individual works. Second, the case for lower marginal tax rates on the poor is weakened by the need to support the incomes of the poor, rather than their welfare, which could be “bought” by allowing them a relatively high amount of leisure: supporting the incomes of the poor calls for relatively high marginal tax rates in the lower part of the income distribution, and the revenue needed for this support requires that sufficiently high average tax rates be imposed on higher incomes. The simulations suggest that these two opposing effects broadly offset one another.

IV. TARGETING BY INDICATORS

In the analysis so far, individuals have been assumed to differ only in their unobserved ability. It is now widely recognized, however, that there are poten-
tially severe incentive and other costs of administering income-related transfers. One way of overcoming these costs, particularly in developing countries, is to differentiate the population by easily observable indicators that are correlated with the unobservable characteristic of interest. An individual's labor market status or demographic attributes, for instance, may convey information on underlying ability. Transfers can usefully be made contingent upon such characteristics. The theory of the optimal use of such information was first considered by Akerlof (1978) and developed by, among others, Kanbur (1987), Besley and Kanbur (1988), and Ravallion (1987). But most of the simple rules of thumb for targeting that have been developed simply assume away labor supply effects. An exception is Kanbur and Keen (1989), who develop a relatively simple framework that gives some feel for the optimal use of nonincome information in the presence of incentive effects. This section reports on that work.

Suppose the population can be divided into two mutually exclusive and exhaustive groups, A and B. The underlying contingencies are assumed to be absolute, so households are unable to switch between groups. The contingency is costlessly verifiable, but we assume—to keep matters simple—that only linear income taxation is feasible. What makes the problem interesting is that distinct schedules may be applied to the two groups: they may be faced, that is, with different poll subsidies $G_K$ and with different marginal tax rates $t_K$ (for $K = A, B$). This ability to treat the two groups differently is only valuable, of course, if they differ in some way that is relevant for poverty alleviation. We allow them to differ in two respects. First, the within-group wage distributions $g_K(w)$ may differ. Thus one group may, for instance, be systematically poorer than the other. Second, they may differ in the responsiveness of their labor supply behavior.

Specifically, we assume that although all individuals have Cobb-Douglas preferences, as in equation 12, the parameter $\alpha$ may differ across the two groups. Imposing the further restriction, for definiteness, that poverty is to be assessed in terms of the Foster-Greer-Thorbecke index, the objective of policy is thus taken to be the minimization (subject to the government's budget constraint) of

$$P^* = \Theta P^*_A + (1 - \Theta) P^*_B$$

where $P^*_K$ is defined as in equation 13, $\theta$ is the proportion of the population in group A, and the net income of a type $K$ household with pretax wage $w$ is

$$x(w; K) = (1 - \delta_K) [(1 - t_K) w + G_K].$$

The two groups are assumed to have the same poverty line, $z$. This precludes a range of (troubling) issues concerning the relation between needs and optimally targeted benefits. Depending on the form of the deprivation function, $D(\cdot)$, it may be, for example, that the level of support optimally targeted to a group varies inversely with its neediness, as measured by $z$. The intuition is that the very needy may simply be too expensive to help (see Keen 1992).
Taking the tax rates $t_K$ as given, under what circumstances would aggregate poverty $P^e$ be reduced by cutting the poll subsidy given to one group in order to finance an increase in that paid to the other? A retargeting of support of this kind away from group B and toward group A can be shown to reduce aggregate poverty if and only if

$$\sigma(\delta_A, t_A) P_A^{\alpha-1} > \sigma(\delta_B, t_B) P_B^{\alpha-1}$$

where

$$\sigma(\delta, t) = \frac{(1 - \delta)(1 - t)}{1 - t(1 - \delta)}.$$ 

To develop the intuition behind inequality 16, consider first the role of the $P_K^{-1}$ terms. These emphasize the simple but important point that the reduction of aggregate poverty measured in some particular way is typically not best pursued by redirecting resources toward whichever group is poorest in terms of that same measure. What matters is the marginal effect on the measure of interest. The structure of the $P^e$ index happens to be such that the implied rule takes an especially simple form. Assuming away incentive effects for the moment, so that $\delta_A = \delta_B = 0$, minimization of the aggregate index for some specific choice of $\alpha$ requires looking first at the within-group indexes for $\alpha - 1$. Suppose, for instance, that we have chosen $\alpha = 1$. This means that our objective is simply to minimize the aggregate poverty gap, or, equivalently, to maximize the net income of the poor. Imagine now that we have some fixed sum to spend on increasing the poll subsidy $G_K$, to one group or the other (and assume for simplicity that $\theta = 1/2$). Which group should we favor? The disadvantage of having to spend this money as a poll subsidy is that some of it will be wasted on the nonpoor; giving it only to group K, the proportion of our fixed sum that will reach the poor is just the proportion of that group that is in poverty. To achieve the largest possible increase in the total income of the poor, we should therefore allocate the funds to whichever group has the larger number of poor individuals—that is, to whichever group has the higher $P^e_K$.

Incentive effects enter the story through the $\sigma(\cdot)$ terms in inequality 16, with retargeting toward group A more likely to be desirable, other things being equal, the higher $\sigma(\delta_A, t_A)$ is and the lower $\sigma(\delta_B, t_B)$ is. It is easily seen from equation 17 that $\sigma(\delta, t)$ is decreasing in both $\delta$ and $t$. Thus group A is more likely to be favored the less responsive its labor supply behavior is and the lower the marginal tax rate is that it initially faces. The intuition is straightforward. When $\delta_A$ is relatively low, the income effect of increasing the poll subsidy to group A—which points toward a reduction in hours and hence in net income, dampening the beneficial impact on poverty—is relatively weak. Conversely, a high $\delta_B$ indicates a relatively powerful income effect acting to mitigate the impact of reduc-

8. Proofs of the claims that follow are in Kanbur and Keen (1989).
ing the poll subsidy to group B. And when \( t_A \) is relatively low, so too is the revenue cost of the reduction in hours worked—and hence taxes paid—by members of group A as a result of their higher lump-sum income. Conversely, a high \( t_B \) is helpful in recouping revenue from the increased labor supply of group B.

The tension to which inequality 16 points emerges especially clearly if the initial position is one in which \( t_A = t_B = 0 \). Retargeting toward A is then desirable if and only if

\[
(1 - \delta_A) P_A^{\alpha-1} > (1 - \delta_B) P_B^{\alpha-1}.
\]

We have already discussed why the natural inclination to favor the group with the higher incidence of poverty has to be modified to favor the group with higher \( P_A^{\alpha-1} \). But incentive effects can more than offset this consideration. It may be optimal to cut the poll subsidy paid to the group with the higher \( P_A^{\alpha-1} \) if its labor supply behavior is sufficiently more sensitive (that is, if \( \delta \) for that group is sufficiently high).

The targeting rule expressed in inequality 16 is valid for arbitrary marginal tax rates \( t_K \). When these, too, can be chosen by the government, from the associated first-order conditions, poverty minimization requires that

\[
\frac{P_A^{\alpha-1} - P_A}{P_B^{\alpha-1} - P_B} = \frac{\bar{x}_A}{\bar{x}_B}
\]

where \( \bar{x}_K \) denotes the mean net income of group K. The difference between the \( P_A \) and \( P_B^{\alpha-1} \) indexes must thus stand in the same ratio across groups as do their mean net incomes. For the case in which \( \alpha = 1 \), this reduces to the simple condition that

\[
\Gamma_A = \Gamma_B
\]

where \( \Gamma_K \) denotes the share of the poor in group K of the total net income of that group. The significance of the rules expressed in equations 19 and 20 is less in the additional insight they convey—which adds little to what has gone before—than in their applicability. They show how simplifying assumptions can be used to incorporate labor supply responses, in a relatively straightforward way, into the use of indicators for targeting.

V. Conclusion

Labor supply introduces some new considerations into the design of poverty alleviation programs. First and foremost, it forces us to reconsider the standard objective function according to which these programs are evaluated: the minimization of poverty as measured by the shortfall of income or expenditure from a critical value. This objective leaves out of consideration the effort that individuals make in earning their incomes. How is this effort, or rather the leisure that is lost in making it, to be valued? Valuing it at the market wage—which is the

9. Details of the proof can be found in Kanbur and Keen (1989).
A good example of how the new perspective can alter basic results in the targeting literature is provided in the section on targeting by commodities. Besley and Kanbur (1988) establish, under certain conditions, the validity of the simple rule of thumb that commodity subsidies should focus on those commodities whose consumption by the poor is a large fraction of total consumption. This is done in a welfaristic framework. However, if labor supply is elastic, then, under the nonwelfarist approach considered here, the rule is modified depending on the weight given to disutility of effort in evaluating poverty. If no weight is given at all, then the case for subsidizing good \( i \) is weaker the more such a subsidy would tend to reduce labor supply, that is, the greater the complementarity between \( i \) and leisure. But this result is reversed by attaching a sufficiently high weight to the disutility of effort. It is then no longer acceptable to provide consumption at the poverty line by inducing individuals to work excessively, and complements with leisure should therefore be subsidized rather than taxed.

The rest of the article followed through the consequences of assuming that no weight is given to leisure in the social welfare function. For income-based targeting (and for conventional parameter values), the optimal marginal withdrawal of benefits as income increases is around 50 to 60 percent. This should provide a benchmark for the evaluation of income-tested schemes. Marginal withdrawal rates far above this may look good from the simplest targeting perspective, but the incentive effects are liable to dominate any targeting gains. Finally, we considered modifications to rules of thumb in non-income-based targeting. We showed that for any indicator that divides the population into mutually exclusive groups for targeting purposes, positive correlation between labor supply elasticity and poverty incidence across the groups reduces the usefulness of the indicator. Thus, relying only on poverty incidence can give a false sense of the value of an indicator for targeting purposes.

This article is only a start in the direction of introducing labor supply considerations into the targeting of poverty alleviation programs in developing countries. We end by noting that the issues raised here extend well beyond the specific case of labor supply and income poverty. They apply to any measure of the standard of living (such as nutrition) when individuals have choices to make between alternative forms of consumption and differ in their ability to transform one type of consumption into another.
Appendix. The Derivation of Equation 9

We describe here the derivation of equation 9. Using the government’s budget constraint

\[ R = t_1 \int_0^\infty x_1(q, w) g(w) \, dw + t_2 \int_0^\infty x_2(q, w) g(w) \, dw \]  

(A-1)

to define \( t_2 \) as a function of \( t_1 \) (for fixed \( R \)), the effect on poverty—defined as in equation 7—of slightly increasing the tax on good 1 in order to lower that on good 2 is given by

\[ \frac{dP}{dt_1} = \frac{\partial P}{\partial t_1} + \frac{\partial P}{\partial t_2} \frac{dt_2}{dt_1} \mid R. \]  

(A-2)

Differentiating A-1 at \( t_1 = t_2 = 0 \) gives

\[ \frac{dt_2}{dt_1} \mid R = -\frac{\bar{x}_1}{\bar{x}_2} \]  

(A-3)

and from equation 7, and assuming that \( D(z, z) = 0 \),

\[ \frac{\partial P}{\partial t_2} = Dy \frac{\partial y}{\partial q_1} g(w) \, dw. \]  

(A-4)

Assuming constant producer prices, \( p \), equation 6 implies that

\[ \frac{\partial y}{\partial q_1} = \bar{x} \frac{\partial x}{\partial q_1} - s^h L \frac{\partial L}{\partial q_i}, \]  

(A-5)

\[ = -\left[ x_i + (s^h_i - \omega) \frac{\partial L}{\partial q_i} \right]. \]  

(A-6)

Equation A-6 follows from the choice \( s_x = p \), the assumption that both taxes are initially zero (so that \( p = q \)), and differentiation of the individual’s budget constraint, \( q \cdot x - \omega L = 0 \). Substituting equations A-3 to A-6 in equation A-1 gives equation 9.

References

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